PLASMA KINETIC THEORY

Chap. 1 Physical Characteristics and Math Description of Waves in Plasma
波动的物理特点和数学描述

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Contents

主要参考自胡希伟老师的教材《等离子体理论基础》、马锦秀老师的课堂讲义。本稿尚不完善，还会更新。

1.1 Vlasov-Maxwell Equations 方程组

1.1.1 Linear Problem and Classification

线性问题及分类

$$f_\alpha = f_\alpha(\boldsymbol{r}, \boldsymbol{v}, t) \tag{1}$$

$$n_\alpha(\boldsymbol{r}, t) = \int f_\alpha(\boldsymbol{r}, \boldsymbol{v}, t) \overbrace{\mathrm{d}\boldsymbol{v}}^{\mathrm{d}x\mathrm{d}y\mathrm{d}z} \tag{2}$$

$$N_\alpha = \int n_\alpha(\boldsymbol{r}, t) \overbrace{\mathrm{d}\boldsymbol{r}}^{\mathrm{d}v_x\mathrm{d}v_y\mathrm{d}v_z} = \iint f_\alpha(\boldsymbol{r}, \boldsymbol{v}, t)\mathrm{d}\boldsymbol{r}\mathrm{d}\boldsymbol{v} \tag{3}$$

$$\text{归一化分布函数：} \hat{f}_\alpha = f_\alpha / n_\alpha \quad s.t. \quad \int \hat{f}_\alpha \mathrm{d}\boldsymbol{v} = 1 \tag{4}$$

空间均匀：n_α 为常数，$\hat{f}_\alpha$ 代表归一化的速度分布。

空间非均匀：n_α 与 $\boldsymbol{r}$ 有关，$\hat{f}_\alpha$ 代表局域分布几率。

f_α 满足 Boltzmann 方程（采用 cgs Gauss 单位制）

$$\partial_t f_\alpha + \boldsymbol{v} \cdot \nabla_{\boldsymbol{r}} f_\alpha + \frac{q_\alpha}{m_\alpha}\left(\boldsymbol{E} + \frac{\boldsymbol{v}}{c} \times \boldsymbol{B}\right) \cdot \nabla_{\boldsymbol{v}} f_\alpha = (\partial_t f_\alpha)_c \tag{5}$$

$\boldsymbol{E}$ 和 $\boldsymbol{B}$ 满足 Maxwell 方程组，下面 ρ_{ext}, $\boldsymbol{J}_{ext}$ 表示等离子体系统外的其他电荷和电流密度。

$$\begin{cases} \nabla \cdot \boldsymbol{E} = 4\pi \sum_\alpha q_\alpha \int f_\alpha \mathrm{d}\boldsymbol{v} + 4\pi \rho_{ext} \\ \nabla \times \boldsymbol{E} = -\frac{1}{c}\partial_t \boldsymbol{B} \\ \nabla \cdot \boldsymbol{B} = 0 \\ \nabla \times \boldsymbol{B} = \frac{1}{c}\partial_t \boldsymbol{E} + \frac{4\pi}{c} \sum_\alpha q_\alpha \int \boldsymbol{v} f_\alpha \mathrm{d}\boldsymbol{v} + \frac{4\pi}{c}\boldsymbol{J}_{ext} \end{cases} \tag{6}$$

$$\text{电荷密度：}\ \rho(\boldsymbol{r}, t) = \sum_\alpha \rho_\alpha = \sum_\alpha q_\alpha n_\alpha = \sum_\alpha q_\alpha \int f_\alpha \mathrm{d}\boldsymbol{v} \tag{7}$$

$$\text{电流密度：}\ \boldsymbol{J}(\boldsymbol{r}, t) = \sum_\alpha \boldsymbol{J}_\alpha = \sum_\alpha q_\alpha \boldsymbol{\Gamma}_\alpha = \sum_\alpha q_\alpha \int \boldsymbol{v} f_\alpha \mathrm{d}\boldsymbol{v} \tag{8}$$

波动 -> 小扰动（主要指线性的波动，非线性波动不算）

物理量：分为平衡态量（零阶量，下标为 0）和扰动量（下标为 1）。

$$f_\alpha = f_{\alpha 0} + f_{\alpha 1}, \quad \boldsymbol{E} = \boldsymbol{E}_0 + \boldsymbol{E}_1, \quad \boldsymbol{B} = \boldsymbol{B}_0 + \boldsymbol{B}_1 \tag{9}$$

【注意：$\boldsymbol{r}, \boldsymbol{v}$ 为相空间中的自变量，$\boldsymbol{v}$ 不是流体力学中的微元速度 $\boldsymbol{u}$】

$$|f_{\alpha 1}| \ll f_{\alpha 0}, |\boldsymbol{E}_1| \ll \boldsymbol{E}_0 \text{ 或}\boldsymbol{E}_1\text{很小}, |\boldsymbol{E}_1| \ll \boldsymbol{E}_0 \text{ 或}\boldsymbol{E}_1\text{很小}, \tag{10}$$

平衡态（零阶）方程组：

$$\partial_t f_{\alpha 0} + \boldsymbol{v} \cdot \nabla_{\boldsymbol{r}} f_{\alpha 0} + \frac{q_\alpha}{m_\alpha}(\boldsymbol{E}_0 + \frac{\boldsymbol{v}}{c} \times \boldsymbol{B}_0) \cdot \nabla_{\boldsymbol{v}} f_{\alpha 0} = (\partial_t f_{\alpha 0})_c \tag{11}$$

$$\begin{cases} \nabla \cdot \boldsymbol{E}_0 = 4\pi \sum_\alpha q_\alpha \int f_{\alpha 0} \mathrm{d}\boldsymbol{v} \\ \nabla \times \boldsymbol{E}_0 = -\frac{1}{c}\partial_t \boldsymbol{B}_0 \\ \nabla \cdot \boldsymbol{B}_0 = 0 \\ \nabla \times \boldsymbol{B}_0 = \frac{1}{c}\partial_t \boldsymbol{E}_0 + \frac{4\pi}{c}\sum_\alpha q_\alpha \int \boldsymbol{v} f_{\alpha 0} \mathrm{d}\boldsymbol{v} + \frac{4\pi}{c}\boldsymbol{J}_{ext} \end{cases} \tag{12}$$

线性化的扰动方程组（一阶）：

$$\partial_t f_{\alpha 1}+\boldsymbol{v}\cdot\nabla_{\boldsymbol{r}} f_{\alpha 1}+\frac{q_\alpha}{m_\alpha}(\boldsymbol{E}_1+\frac{\boldsymbol{v}}{c}\times\boldsymbol{B}_1)\cdot\nabla_{\boldsymbol{v}} f_{\alpha 0}+\frac{q_\alpha}{m_\alpha}(\boldsymbol{E}_0+\frac{\boldsymbol{v}}{c}\times\boldsymbol{B}_0)\cdot\nabla_{\boldsymbol{v}} f_{\alpha 1}=(\partial_t f_{\alpha 1})_c \tag{13}$$

$$\begin{cases} \nabla\cdot\boldsymbol{E}_1=4\pi\sum_\alpha q_\alpha\int f_{\alpha 1}\mathrm{d}\boldsymbol{v}+4\pi\rho_{ext} \\ \nabla\times\boldsymbol{E}_1=-\frac{1}{c}\partial_t\boldsymbol{B}_1 \\ \nabla\cdot\boldsymbol{B}_1=0 \\ \nabla\times\boldsymbol{B}_1=\frac{1}{c}\partial_t\boldsymbol{E}_1+\frac{4\pi}{c}\sum_\alpha q_\alpha\int\boldsymbol{v}f_{\alpha 1}\mathrm{d}\boldsymbol{v} \end{cases} \tag{14}$$

常用热平衡分布：Maxwell 分布，在均匀、各向同性、无外场的情况下

$$
\begin{aligned}
f_{\alpha M}(\boldsymbol{v}) =& n_{\alpha 0}\left(\frac{m_\alpha}{2\pi T_\alpha}\right)^{3/2} e^{-\frac{m_\alpha}{2T_\alpha}(\boldsymbol{v}-\boldsymbol{u}_\alpha)^2}, && \boldsymbol{u}_\alpha \text{ 宏观流动速度} \\
=& n_{\alpha 0}\frac{1}{\pi^{3/2} v_{t\alpha}^3} e^{-\frac{(\boldsymbol{v}-\boldsymbol{u}_\alpha)^2}{v_{t\alpha}^2}}, && \boldsymbol{v}_{t\alpha} = \left(\frac{2T_\alpha}{m_\alpha}\right)^{1/2} \text{ 热速度}
\end{aligned}
\tag{15}
$$

线性波动问题分类

根据平衡态可划分具体的情况：

1) 无外场 $\boldsymbol{B}_0 \neq \mathbf{0}$，空间均匀 $\nabla_{\boldsymbol{r}} f_{\alpha 0} = 0$，速度空间各向同性 $f_{\alpha 0}(\boldsymbol{v}) = f_{\alpha 0}(v^2)$，无碰撞 $(\partial_t f_\alpha)_c = 0$ 情况下的静电波（ES）和电磁波（EM）。
2) 均匀恒定磁场 $\boldsymbol{B}_0 = const.$，空间均匀 $\nabla_{\boldsymbol{r}} f_{\alpha 0} = 0$，速度空间各向同性或异性 $f_{\alpha 0}(\boldsymbol{v})$，无碰撞 $(\partial_t f_\alpha)_c = 0$ 情况下的静电波（ES）和电磁波（EM）。
3) 均匀恒定磁场 $\boldsymbol{B}_0 = const.$，空间不均匀 $\nabla_{\boldsymbol{r}} f_{\alpha 0} \neq 0$，速度空间各向异性 $f_{\alpha 0}(\boldsymbol{v})$，无碰撞 $(\partial_t f_\alpha)_c = 0$ 情况下的漂移波。
4) 非均匀恒定磁场 $\boldsymbol{B}_0 \neq const.$，空间不均匀 $\nabla_{\boldsymbol{r}} f_{\alpha 0} \neq 0$，速度空间各向异性 $f_{\alpha 0}(\boldsymbol{v})$，有碰撞 $(\partial_t f_\alpha)_c \neq 0$ 情况下的耗散波。

线性波动问题的种类

1) 本征模，固有的集体运动模式，特点 $\omega = \omega(\boldsymbol{k})$ 对所有粒子 ω 相同。
2) 初值问题相关的弹道模（ballistic mode）特点 $\omega = \boldsymbol{k} \cdot \boldsymbol{v}$ 不同 $\boldsymbol{v}$ 的粒子 ω 不同。
   对初始扰动的响应：包括本征模和弹道模。
3) 参量过程，对外加持续驱动的响应。

对于无边界的无限空间，采用 Fourier 变换，$A(\boldsymbol{r}) \rightleftharpoons A(\boldsymbol{k})$

$$\text{正变换：} A(\boldsymbol{k}) = \frac{1}{(2\pi)^3}\int_{-\infty}^{\infty} A(\boldsymbol{r}) e^{-i\boldsymbol{k}\cdot\boldsymbol{r}} \mathrm{d}\boldsymbol{r} \tag{16}$$

$$\text{逆变换：} A(\boldsymbol{r}) = \int_{-\infty}^{\infty} A(\boldsymbol{k}) e^{i\boldsymbol{k}\cdot\boldsymbol{r}} \mathrm{d}\boldsymbol{k} \tag{17}$$

对于时间，采用 Laplace 变换 $A(t) \rightleftharpoons A(p)$.

$$\text{正变换：} A(p) = \int_0^{\infty} A(t) e^{-pt} \mathrm{d}t, \quad p \in \mathbb{C}, \mathrm{Re}(p) \geq p_0 \geq 0 \text{时使积分收敛。} \tag{18}$$

$A'(t)$ 的 Laplace 变换与 $A(t)$ 本身有什么关系？

$$\begin{aligned}\mathcal{L}\left\{A'(t)\right\}(p) &= \int_0^{\infty} \frac{dA}{dt} e^{-pt} dt = \int_0^{\infty} \frac{d}{dt}\left(Ae^{-pt}\right) dt + \int_0^{\infty} pAe^{-pt} dt \\ &= pA(p) - A(t=0)\end{aligned} \tag{19}$$

Laplace 逆变换为复数平面路径积分：

$$A(t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} A(p)e^{pt}\mathrm{d}p, \quad c \geq p_0, \tag{20}$$

令 $\omega = ip$, 使逆变换与 Fourier 变换的形式一致，表示一列波，$p = -i\omega$ 正变换：

$$A(\omega) = \int_0^t A(t)e^{i\omega t}\mathrm{d}t, \quad \mathrm{Im}(\omega) = \gamma \geq \gamma_0 \geq 0\text{使积分收敛} \tag{21}$$

$A'(t)$ 的变换：

$$A'(\omega) = -i\omega A(\omega) - A(0) \tag{22}$$

逆变换：

$$
\begin{aligned}
A(t) &= \frac{1}{2\pi i}\int_{ic+\infty}^{ic-\infty} A(\omega)e^{-\omega t}\mathrm{d}(-i\omega) \\
&= \frac{1}{2\pi}\int_{-\infty+ic}^{\infty+ic} A(\omega)e^{-\omega t}\mathrm{d}\omega, c \geq \gamma_0
\end{aligned}
\tag{23}
$$

对于本征值问题，无时间起点，$\Rightarrow$ Fourier 变换

解析延拓：

$$A(\omega) = \int_{-\infty}^{\infty} A(t)e^{i\omega t}\mathrm{d}t = \left(\int_{-\infty}^{0} + \int_{0}^{\infty}\right) A(t)e^{i\omega t}\mathrm{d}t \tag{24}$$

其中，$\int_0^\infty$ 要求 $\gamma \geq \gamma_0 \geq 0$ 使之收敛；$\int_{-\infty}^0$ 要求 $\gamma \leq \gamma_0' \leq 0$ 使之收敛，两者矛盾。

要求 $t \to -\infty$ 过程中 $A(t) \to 0$ 比 $e^{i\omega t} \to \infty$ 快。

即要求：$\gamma \geq \gamma \gtrsim 0$ 并且 $t \to -\infty, A(t)e^{i\omega t} \to 0$ 使积分收敛，即 $t \to \infty$ 和 $t \to -\infty$ 时波动均为小扰动

则逆变换

$$A(t) = \frac{1}{2\pi}\int_{-\infty+i0^+} A(\omega)e^{-i\omega t}\mathrm{d}\omega \tag{25}$$

波的表达形式

一列波，其某一物理量分量变化形式如 $A(\boldsymbol{r},t)=\bar{A}e^{i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)}$, $\bar{A}\in\mathbb{C}$.

角频率和波矢均可复数分解实部、虚部：$\omega=\omega_r+i\gamma$, $\boldsymbol{k}=\boldsymbol{k}_r+i\boldsymbol{k}_i$.

（1）若 ω 虚部非零：

考虑一维 $\boldsymbol{k}=k\hat{\mathbf{e}}_x$ 简单些, $A(\boldsymbol{r},t)=\bar{A}e^{i(kx-\omega_r t)}e^{\gamma t}=\underbrace{\bar{A}e^{\gamma t}}_{\text{振幅}}\underbrace{e^{i(kx-\omega_r t)}}_{\text{相位}}$.

相速度：$v_\phi=\omega_r/k$

群速度：$v_g=\mathrm{d}\omega_r/\mathrm{d}k$

幅度变化：$\gamma>0$ 即增长，$\gamma<0$ 即阻尼。

（2）若 $\boldsymbol{k}$ 虚部非零，ω 为实数, $A(\boldsymbol{r},t)=\underbrace{\bar{A}e^{-k_i x}}_{\text{振幅}}\underbrace{e^{i(k_r x-\omega t)}}_{\text{相位}}$

幅度变化：$k_i>0$ 即衰减，$k_i<0$ 即放大。

波的表达形式

（3）若 $\omega, \boldsymbol{k}$ 均为复数，$A(\boldsymbol{r}, t) = \underbrace{\bar{A}e^{-k_i x+\gamma t}}_{\text{振幅}} \underbrace{e^{i(k_r x-\omega_r t)}}_{\text{相位}}$

对等离子体中的波，若 $\gamma > 0$ 则称为不稳定性。

根据其群速度还可分为：

- 绝对不稳定性（absolute instability）$\boldsymbol{v}_g = \mathbf{0}$
- 随流不稳定性（convective instability）$\boldsymbol{v}_g \neq \mathbf{0}$

1.2 Energy Propaation Equation of Perturbation Wave Field 扰动波场的能量传播方程

1.2.1 Poynting Theorem 坡印廷定理

扰动电磁场、电流密度：$\boldsymbol{E}_1$、$\boldsymbol{B}_1$、$\boldsymbol{J}_1$

$$\nabla \times \boldsymbol{E}_1 = -\frac{1}{c}\frac{\partial \boldsymbol{B}_1}{\partial t}, \qquad \nabla \times \boldsymbol{B}_1 = \frac{4\pi}{c}\boldsymbol{J}_1 + \frac{1}{c}\frac{\partial \boldsymbol{E}_1}{\partial t}$$

$$\partial_t w + \nabla \cdot \boldsymbol{p} + \boldsymbol{E}_1 \cdot \boldsymbol{J}_1 = 0 \tag{26}$$

$$\begin{aligned} w &= \frac{1}{8\pi}(\boldsymbol{E}_1^2 + \boldsymbol{B}_1^2) && \text{电磁场能量密度} \\ \boldsymbol{P} &= \frac{c}{4\pi}(\boldsymbol{E}_1 \times \boldsymbol{B}_1) && \text{Poynting 矢量, 电磁场能流密度} \\ & \boldsymbol{E}_1 \cdot \boldsymbol{J}_1 && \text{介质能耗} \end{aligned} \tag{27}$$

介质响应（欧姆定律）

静态或准静态：$\boldsymbol{J}_1 = \sigma \boldsymbol{E}_1$, σ 即（各向同性时）电导率标量

单色平面波（频谱单一）：

$$\boldsymbol{J}_1(\boldsymbol{k},\omega) = \boldsymbol{\sigma}(\boldsymbol{k},\omega)\cdot\boldsymbol{E}_1(\boldsymbol{k},\omega) \tag{28}$$

一般情况：

$$\begin{aligned}\boldsymbol{J}_1(\boldsymbol{r},t) &= \iint \boldsymbol{\sigma}(\boldsymbol{r}',t)\cdot\boldsymbol{E}_1(\boldsymbol{r}-\boldsymbol{r}',t-t')\mathrm{d}\boldsymbol{r}'\mathrm{d}t' \\ &= \iint \boldsymbol{\sigma}(\boldsymbol{r}-\boldsymbol{r}',t-t')\cdot\boldsymbol{E}_1(\boldsymbol{r}',t)\mathrm{d}\boldsymbol{r}'\mathrm{d}t',\ \text{(nonlocal effect)}\end{aligned} \tag{29}$$

准单色平面波

讨论准单色平面波时，我们指 $\Delta \sim \gamma \ll \omega_r$，电场包络幅值慢变的情况，如下式 $\boldsymbol{E}_0(t)$ 即慢变振幅，需 $|\partial_t \boldsymbol{E}_0| \ll |\omega_0 \boldsymbol{E}_0|$

$$\boldsymbol{E}_1(t) = \frac{1}{2}\boldsymbol{E}_0(t)e^{-i\omega_0 t} + \frac{1}{2}\boldsymbol{E}_0^*(t)e^{i\omega_0 t} \tag{30}$$

$$\mathcal{F}.T. \quad \boldsymbol{E}_1(\omega) = \int \boldsymbol{E}_1(t)e^{i\omega t}dt = \frac{1}{2}\boldsymbol{E}_0\left(\omega - \omega_0\right) + \frac{1}{2}\boldsymbol{E}_0^*\left(\omega + \omega_0\right) \tag{31}$$

电流密度：

$$\boldsymbol{J}_1(\omega) = \boldsymbol{\sigma}(\omega)\cdot\boldsymbol{E}_1(\omega) = \frac{1}{2}\boldsymbol{\sigma}(\omega)\cdot\boldsymbol{E}_0\left(\omega - \omega_0\right) + \frac{1}{2}\boldsymbol{\sigma}(\omega)\cdot\boldsymbol{E}_0^*\left(\omega + \omega_0\right) \tag{32}$$

反变换：

$$\begin{aligned}
\boldsymbol{J}_1(t) &= \int \boldsymbol{J}_1(\omega)e^{-i\omega t}\frac{d\omega}{2\pi} \\
&= \frac{1}{2}\int \boldsymbol{\sigma}(\omega)\cdot\boldsymbol{E}_0\left(\omega-\omega_0\right)e^{-i\omega t}\frac{d\omega}{2\pi} + \frac{1}{2}\int \boldsymbol{\sigma}(\omega)\cdot\boldsymbol{E}_0^*\left(\omega+\omega_0\right)e^{-i\omega t}\frac{d\omega}{2\pi} \\
&= \frac{1}{2}\int \boldsymbol{\sigma}\left(\omega_0+\omega'\right)\cdot\boldsymbol{E}_0\left(\omega'\right)e^{-i\left(\omega_0+\omega'\right)} \\
&\quad + \boldsymbol{\sigma}\left(-\omega_0+\omega'\right)\cdot\boldsymbol{E}_0^*\left(\omega'\right)e^{-i\left(-\omega_0+\omega'\right)}\frac{d\omega}{2\pi}
\end{aligned} \tag{33}$$

则

$$\boldsymbol{\sigma}\left(\pm\omega_0+\omega'\right) \cong \boldsymbol{\sigma}\left(\pm\omega_0\right) + \left(\frac{\partial\boldsymbol{\sigma}}{\partial\omega}\right)_{\omega=\pm\omega_0}\cdot\omega' \tag{34}$$

$$\Rightarrow \quad \boldsymbol{J}_1(t) = \frac{1}{2}\boldsymbol{J}_0(t)e^{-i\omega_0 t} + \frac{1}{2}\boldsymbol{J}_0^*(t)e^{i\omega_0 t} \tag{35}$$

$$\Rightarrow \boldsymbol{J_0}(t) = \boldsymbol{\sigma}\left(\omega_0 + i\frac{\partial}{\partial t}\right)\cdot \boldsymbol{E}_0(t), \quad \boldsymbol{J_0^*}(\boldsymbol{t}) = \boldsymbol{\sigma}\left(-\omega_0 + i\frac{\partial}{\partial t}\right)\cdot \boldsymbol{E}_0^*(t) \tag{36}$$

$\boldsymbol{J}_1(t)$ 为实数：

$$\begin{aligned} \boldsymbol{J}_1^*(t) = \boldsymbol{J}_1(t) \Rightarrow \boldsymbol{\sigma} * \left(\omega_0 + i\frac{\partial}{\partial t}\right) &= \boldsymbol{\sigma}\left(-\omega_0 + i\frac{\partial}{\partial t}\right) \\ \boldsymbol{\sigma}^*\left(-\omega_0 + i\frac{\partial}{\partial t}\right) &= \boldsymbol{\sigma}\left(\omega_0 + \frac{i\partial}{\partial t}\right) \end{aligned} \tag{37}$$

1.2.2 Energy Propagation Equation of Wave

波的能量传播方程（波幅方程）

对快变的位相平均，即在一个快变周期内平均

$$
\begin{aligned}
\langle \boldsymbol{E}_1(t)\cdot \boldsymbol{J}_1(t)\rangle &= \frac{1}{4}\boldsymbol{E}_0(t)\cdot \boldsymbol{J}_0^*(t)+\frac{1}{4}\boldsymbol{E}_0^*(t)\cdot \boldsymbol{J}_0(t)\\
&= \frac{1}{4}\left[\boldsymbol{E}_0\cdot\boldsymbol{\sigma}\left(-\omega_0+i\frac{\partial}{\partial t}\right)\cdot\boldsymbol{E}_0^*+\boldsymbol{E}_0^*\cdot\boldsymbol{\sigma}\left(\omega_0+i\frac{\partial}{\partial t}\right)\cdot\boldsymbol{E}_0\right]\\
&= \frac{1}{4}\left\{\boldsymbol{E}_0^*\cdot\left[\boldsymbol{\sigma}\left(\omega_0\right)+\boldsymbol{\sigma}^T\left(-\omega_0\right)\right]\cdot\boldsymbol{E}_0\right\}\\
&\quad+\frac{i}{4}\left\{\boldsymbol{E}_0^*\cdot\frac{\partial\boldsymbol{\sigma}\left(\omega_0\right)}{\partial\omega_0}\cdot\frac{\partial\boldsymbol{E}_0}{\partial t}-\frac{\partial\boldsymbol{E}_0^*}{\partial t}\cdot\frac{\partial\boldsymbol{\sigma}^T\left(-\omega_0\right)}{\partial\omega_0}\cdot\boldsymbol{E}_0\right\}\\
&\cong \frac{1}{4}\left\{\boldsymbol{E}_0^*\cdot\left[\boldsymbol{\sigma}\left(\omega_0\right)+\boldsymbol{\sigma}^{T*}\left(\omega_0\right)\right]\cdot\boldsymbol{E}_0\right\}\\
&\quad+\frac{i}{8}\frac{\partial}{\partial t}\left\{\boldsymbol{E}_0^*\cdot\frac{\partial}{\partial\omega_0}\left[\boldsymbol{\sigma}\left(\omega_0\right)-\boldsymbol{\sigma}^{T*}\left(\omega_0\right)\right]\cdot\boldsymbol{E}_0\right\}
\end{aligned}
\tag{38}
$$

其中 $\boldsymbol{\sigma}^{T*}$ 表示转置、共轭，加起来即为厄米变换。

定义：

$$\begin{aligned}\boldsymbol{\sigma} &= \boldsymbol{\sigma}^h + i\boldsymbol{\sigma}^a \\ \boldsymbol{\sigma}^h &= \frac{1}{2}\left[\boldsymbol{\sigma} + \boldsymbol{\sigma}^{T*}\right] \\ \boldsymbol{\sigma}^a &= \frac{1}{2i}\left[\boldsymbol{\sigma} - \boldsymbol{\sigma}^{T*}\right]\end{aligned} \tag{39}$$

介电函数张量：

$$\boldsymbol{\epsilon}(\omega) = \boldsymbol{I} + \frac{4\pi i}{\omega}\boldsymbol{\sigma}(\omega) = \boldsymbol{\epsilon}^h + i\boldsymbol{\epsilon}^a, \boldsymbol{I} \tag{40}$$

$$\begin{aligned}\boldsymbol{\epsilon}^h &= \frac{1}{2}(\boldsymbol{\epsilon} + \boldsymbol{\epsilon}^{T*}) = \boldsymbol{I} - \frac{4\pi}{\omega}\boldsymbol{\sigma}^a \\ \boldsymbol{\epsilon}^a &= \frac{1}{2i}(\boldsymbol{\epsilon} - \boldsymbol{\epsilon}^{T*}) = \frac{4\pi}{\omega}\boldsymbol{\sigma}^h\end{aligned} \tag{41}$$

$$\Rightarrow \langle \boldsymbol{E}_1(t)\cdot \boldsymbol{J}_1(t)\rangle \cong \frac{1}{4}\left\{\boldsymbol{E}_0^*\cdot\left[\boldsymbol{\sigma}\left(\omega_0\right)+\boldsymbol{\sigma}^{T*}\left(\omega_0\right)\right]\cdot \boldsymbol{E}_0\right\} \quad \text{耗散的能量}$$

$$\text{像电容器存储的能量} \rightarrow \quad +\frac{i}{8}\frac{\partial}{\partial t}\left\{\boldsymbol{E}_0^*\cdot\frac{\partial}{\partial\omega_0}\left[\boldsymbol{\sigma}\left(\omega_0\right)-\boldsymbol{\sigma}^{T*}\left(\omega_0\right)\right]\cdot \boldsymbol{E}_0\right\} \tag{42}$$

$$\langle \partial_t w\rangle = \frac{1}{16\pi}\partial_t\left(\left|\boldsymbol{E}_0\right|^2+\left|\boldsymbol{B}_0\right|^2\right) \tag{43}$$

$$\langle \boldsymbol{P}\rangle = \frac{c}{16\pi}\left(\boldsymbol{E}_0\times\boldsymbol{B}_0^*+\boldsymbol{E}_0^*\times\boldsymbol{B}_0\right) \tag{44}$$

对快变时空平均后的能量方程

$$\partial_t W^h+\nabla\cdot\langle\boldsymbol{P}\rangle+W^a=0 \tag{45}$$

$$
\begin{aligned}
W^h &= \frac{1}{16\pi}\left\{|\boldsymbol{E}_0|^2 + |\boldsymbol{B}_0|^2\right\} - \frac{1}{4}\boldsymbol{E}_0^* \cdot \frac{\partial \boldsymbol{\sigma}^a(\omega_0)}{\partial \omega_0} \cdot \boldsymbol{E}_0 \quad \text{（体系的能量）} \\
&= \frac{1}{16\pi}\left\{|\boldsymbol{B}_0|^2 + \boldsymbol{E}_0^*(\boldsymbol{r}, t) \cdot \frac{\partial}{\partial \omega_0}\left[\omega_0 \boldsymbol{\epsilon}^h(\boldsymbol{k}_0, \omega_0)\right] \cdot \boldsymbol{E}_0(\boldsymbol{r}, t)\right\} \\
W^a &= \frac{\omega_0}{8\pi}\boldsymbol{E}_0^*(\boldsymbol{r}, t) \cdot \boldsymbol{\epsilon}^a(\boldsymbol{k}_0, \omega_0) \cdot \boldsymbol{E}_0(\boldsymbol{r}, t) \quad \text{（体系的能耗）}
\end{aligned}
\tag{46}
$$

静电波能量方程

静电波 $\boldsymbol{B}_1 = \mathbf{0}$, $\boldsymbol{P} = \mathbf{0}$, $\boldsymbol{k} \parallel \boldsymbol{E}_1$ (纵波): $\boldsymbol{\epsilon}(\boldsymbol{k}, \omega) \Rightarrow \epsilon(\boldsymbol{k}, \omega)$.

$$
\begin{aligned}
\partial_t W_{ES}^r + W_{ES}^i &= 0 \\
W_{ES}^r &= \frac{1}{16\pi} |E_0|^2 \frac{\partial}{\partial \omega_0} [\omega_0 \epsilon_r(\omega_0)] \\
W_{ES}^i &= \frac{1}{8\pi} |E_0|^2 \omega_0 \epsilon_i(\omega_0)
\end{aligned}
\tag{47}
$$

$$
\begin{aligned}
&\Rightarrow \partial_t |E_0|^2 \frac{\partial}{\partial \omega_0} [\omega_0 \epsilon_r(\omega_0)] + 2 |E_0|^2 \omega_0 \epsilon_i(\omega_0) = 0 \\
&\Rightarrow |E_0|^2 = |E_0(t=0)|^2 e^{2\gamma t}, \quad \gamma = -\frac{\omega_0 \epsilon_i(\omega_0)}{\frac{\partial}{\partial \omega_0} [\omega_0 \epsilon_r(\omega_0)]}
\end{aligned}
\tag{48}
$$

$\frac{\partial}{\partial \omega_0}(\omega_0 \epsilon_r) > 0$ 正能波, < 0 则负能波。$\epsilon_i > 0$ 正耗散，< 0 则负耗散。两者符号相同时 $\gamma < 0$，一正一负则 $\gamma > 0$。

1.2.3 Energy Equation Averaged both on Time and Space

同时考虑时间和空间平均的能量方程

扰动电场：$\boldsymbol{E}_1(\boldsymbol{r},t)=\frac{1}{2}\boldsymbol{E}_0(\boldsymbol{r},t)e^{i(\boldsymbol{k}_0\cdot\boldsymbol{r}-\omega_0 t)}+\frac{1}{2}\boldsymbol{E}_0^*(\boldsymbol{r},t)e^{-i(\boldsymbol{k}_0\cdot\boldsymbol{r}-\omega_0 t)}$

扰动电流：$\boldsymbol{J}_1(\boldsymbol{r},t)=\frac{1}{2}\boldsymbol{J}_0(\boldsymbol{r},t)e^{i(\boldsymbol{k}_0\cdot\boldsymbol{r}-\omega_0 t)}+\frac{1}{2}\boldsymbol{J}_0^*(\boldsymbol{r},t)e^{-i(\boldsymbol{k}_0\cdot\boldsymbol{r}-\omega_0 t)}$

$$\boldsymbol{J}_1(\boldsymbol{k},\omega)=\boldsymbol{\sigma}(\boldsymbol{k},\omega)\cdot\boldsymbol{E}_1(\boldsymbol{k},\omega) \tag{49}$$

$$\begin{aligned}\Rightarrow \boldsymbol{J}_0(r,t)&=\boldsymbol{\sigma}(\boldsymbol{k}_0-i\nabla_{\boldsymbol{r}},\omega_0+i\partial_t)\cdot\boldsymbol{E}_0(\boldsymbol{r},t)\\ \boldsymbol{J}_0^*(r,t)&=\boldsymbol{\sigma}(-\boldsymbol{k}_0-i\nabla_{\boldsymbol{r}},-\omega_0+i\partial_t)\cdot\boldsymbol{E}_0^*(\boldsymbol{r},t)\end{aligned} \tag{50}$$

对快变时空平均后的能量方程

$$\partial_t W^h+\nabla\cdot\langle\boldsymbol{P}\rangle+W^a=0 \tag{51}$$

$$
\begin{aligned}
W^h &= \frac{1}{16\pi}\left\{|\boldsymbol{B}_0|^2 + \boldsymbol{E}_0^*(\boldsymbol{r},t)\cdot\frac{\partial}{\partial\omega_0}\left[\omega_0\boldsymbol{\epsilon}^h(\boldsymbol{k}_0,\omega_0)\right]\cdot\boldsymbol{E}_0(\boldsymbol{r},t)\right\} \\
W^a &= \frac{\omega_0}{8\pi}\boldsymbol{E}_0^*(\boldsymbol{r},t)\cdot\boldsymbol{\epsilon}^a(\boldsymbol{k}_0,\omega_0)\cdot\boldsymbol{E}_0(\boldsymbol{r},t) \\
\langle\boldsymbol{P}\rangle &= \frac{c}{16\pi}\left\{\boldsymbol{E}_0^*\times\boldsymbol{B}_0 + \boldsymbol{E}_0\times\boldsymbol{B}_0^* - \frac{\omega_0}{c}\frac{\partial}{\partial\boldsymbol{k}_0}\left[\boldsymbol{E}_0^*(\boldsymbol{r},t)\cdot\boldsymbol{\epsilon}^h(\boldsymbol{k}_0,\omega_0)\cdot\boldsymbol{E}_0(\boldsymbol{r},t)\right]\right\}
\end{aligned}
\tag{52}
$$

定义群速度 $\boldsymbol{v}_g$ （不一定与 $\boldsymbol{E}\times\boldsymbol{B}$ 一致）：

$$
\langle\boldsymbol{P}\rangle \equiv W^h\boldsymbol{v}_g \tag{53}
$$

End of Chapter 1
Enjoy plasma!

Bibliography

胡希伟. (2006). 等离子体理论基础. 北京大学出版社.