ANNALS OF PHYSICS: **15**: 79–101 (1961)

The Guiding Center Approximation to Charged Particle Motion*

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The equations governing the guiding center motion of a charged particle in an electromagnetic field are obtained simultaneously and deductively, without considering individually the special geometric situations in which one effect or another occurs alone. The general expression is derived for the guiding center velocity at right angles to the magnetic field **B**. This expression contains five terms arising in the presence of an electric field. They are in addition to the usual "**E** × **B**" drift. Because these terms are unfamiliar objects in the literature on plasmas, they are illustrated by simple examples. Three of the five drifts occur in rotating plasma machines of the Ixion type. One of these three is also shown to be responsible for the Helmholtz instability of a plasma. A fourth one gives the (low frequency) dielectric constant, while the fifth arises if the direction of **B** is time dependent. A detailed geometric picture of the fifth drift is given.

The equation governing the guiding center motion parallel to **B** is also derived for the general time-dependent field. The conditions are discussed under which it can be integrated into the form of an energy integral.

Finally the component of current density perpendicular to **B** in a collisionless plasma is shown to be the current due to the guiding center drift plus the perpendicular component of the curl of the magnetic moment per unit volume. Proofs of this have been given in the past for special cases, such as static fields, $\nabla \times \mathbf{B} = 0$, etc. This proof holds in general, provided conditions for adiabaticity are met. It is also true, but not proven in this paper, that the component of the current density parallel to **B** is the current due to the guiding center velocity parallel to **B** plus the parallel component of the curl of the magnetic moment per unit volume. A proper proof of the parallel component is quite lengthy.

I. INTRODUCTION

The approximate motion of a charged particle in a slowly varying electromagnetic field has been extensively studied by means of the guiding center approximation, (*1–6*) in which the motion is considered as a perturbation of the

* Work was performed under auspices of the U. S. Atomic Energy Commission.

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helical motion in a uniform static magnetic field. The guiding center motion is useful, for example, in plasma physics research and in studying the terrestrial (Van Allen) radiation. The purpose of the present paper is twofold. First, by use of the intuitive picture of the particle gyrating about a small circle whose center ("guiding center") is slowly drifting, a brief and convenient derivation of the guiding center motion is given. This is done without considering separately special cases in which the various guiding-center drifts appear alone. Second, the differential equation for the guiding center motion is applied to several situations which arise in plasma physics and in machines used for controlled thermonuclear research.

II. THE DIMENSIONLESS EQUATION OF MOTION

The smaller the radius of gyration of a particle compared to the size of the system and the shorter the gyration period compared to the variation time of the fields, the more correct becomes the picture of a slowly drifting circle. One way of making these two ratios smaller is to reduce m/e, where m is the rest mass and e the charge of the particle, while holding unchanged the fields and the initial velocity and position of the particle. The idea of varying the unchangeable m/e of a given particle, like an electron, may be disturbing at first. One can of course imagine a series of experiments with a variety of particles. However, it is instructive to show that regarding the m/e of a given particle as variable is mathematically identical with physically possible experiments on the particle. To show this, it is only necessary to scale the equation of motion of a charged particle, and this will be done before proceeding to the guiding center motion. The equation of motion of a charged particle is

$$(mc/e)\ (d/dt)[\mathbf{v}(1 - \beta^2)^{-1/2}] = \mathbf{v} \times \mathbf{B}(\mathbf{r}) + c\mathbf{E}(\mathbf{r}) + (mc/e)\mathbf{g}(\mathbf{r}) \tag{1}$$

where $\mathbf{r}$ is the particle position at time t, $\mathbf{v}$ equals $\dot{\mathbf{r}}$, β^2 equals v^2/c^2, $\mathbf{E}$ and $\mathbf{B}$ are the electric and magnetic fields and $\mathbf{g}$ is the total nonelectromagnetic force per unit mass. Let the initial conditions be that at $t = 0$, $\mathbf{r} = \mathbf{r}_0$ and $\mathbf{v} = \mathbf{v}_0 v_0$, where $\mathbf{v}_0$ is a unit vector in the direction of the initial velocity. To write Eq. (1) in dimensionless form, introduce the dimensionless quantities $\mathfrak{B} = \mathbf{B}(\mathbf{r}, t)/B_0(t)$, $\mathfrak{E} = (mc/P_0B_0)\mathbf{E}(\mathbf{r}, t)$, $\mathfrak{z} = P_0t/mL$, $\mathfrak{R} = \mathbf{r}/L$, and $\mathfrak{G} = (m^2L/P_0^2)\mathbf{g}$, where P_0 equals the initial momentum $mv_0(1 - \beta_0^2)^{-1/2}$, $B_0(t)$ is the magnetic field at a typical point at time t, and L is a characteristic dimension of the system. In terms of the dimensionless quantities Eq. (1) becomes

$$\frac{P_0\, c}{eB_0\, L}\left\{\frac{d}{d\mathfrak{z}}\left\{\frac{d\mathfrak{R}}{d\mathfrak{z}}\left[1 - \left(\frac{P_0}{mc}\frac{d\mathfrak{R}}{d\mathfrak{z}}\right)^2\right]^{-1/2}\right\} - \mathfrak{G}\right\} = \frac{d\mathfrak{R}}{d\mathfrak{z}} \times \mathfrak{B} + \mathfrak{E} \tag{2}$$

with the initial conditions that at $\mathfrak{z} = 0$, $\mathfrak{R} = \mathbf{r}_0/L$ and $d\mathfrak{R}/d\mathfrak{z} = \mathbf{v}_0[1 + (P_0/mc)^2]^{-1/2} = \mathbf{v}_0 \times$ rest energy/initial total energy. For a given $\mathbf{r}_0/L$

and $\mathbf{v}_0$, the equation of motion (2) and associated initial conditions contain the two dimensionless parameters P_0c/eB_0L and $P_0/mc = (v_0/c)(1 - \beta_0^2)^{-1/2}$. In this paper only particles of nonrelativistic energies[1] will be considered. In the nonrelativistic limit $(P_0/mc)^2 \ll 1$ and Eq. (2) becomes

$$\frac{mcv_0}{eB_0\,L}\left[\frac{d^2\mathfrak{R}}{d\mathfrak{z}^2} - \mathfrak{G}(\mathfrak{R}, \mathfrak{z})\right] = \frac{d\mathfrak{R}}{d\mathfrak{z}} \times \mathfrak{B}(\mathfrak{R}, \mathfrak{z}) + \mathfrak{E}(\mathfrak{R}, \mathfrak{z}) \qquad (3)$$

with the initial conditions that at $\mathfrak{z} = 0$, $\mathfrak{R} = \mathbf{r}_0/L$ and $d\mathfrak{R}/d\mathfrak{z} = \hat{\nu}_0$.

The solution $\mathfrak{R}(\mathfrak{z})$ of the differential Eq. (3) depends only on the magnitude of mcv_0/eB_0L for given dimensionless initial conditions $\mathbf{r}_0/L$ and $\hat{\nu}_0$, and for given dimensionless fields $\mathfrak{B}(\mathfrak{R}, \mathfrak{z})$, $\mathfrak{E}(\mathfrak{R}, \mathfrak{z})$, and $\mathfrak{G}(\mathfrak{R}, \mathfrak{z})$. Therefore if mcv_0/eB_0L can be decreased by decreasing v_0/B_0L instead of m/e, and at the same time maintaining $\mathfrak{B}$, $\mathfrak{E}$, $\mathfrak{G}$, $\mathbf{r}_0/L$, and $\mathbf{v}_0$ unchanged, one has a physically possible experiment. Three basic ways of doing this will now be described, along with the necessary changes in the quantities in $\mathbf{B}$, $\mathbf{E}$, $\mathbf{g}$, $\mathbf{r}_0$, and $\mathbf{v}_0$. It is assumed that the experimenter has these quantities, the time scale, and dimensions of the system under his control.

Firstly, suppose that from one experiment to the next $B_0(t)$ is increased by a factor $f > 1$ at all t, while v_0 and L remain unchanged. Since $\mathfrak{B} = \mathbf{B}/B_0$, and since $\mathfrak{B}$ is to be unchanged, the magnetic field $\mathbf{B}(\mathbf{r}, t)$ must be multiplied by f at all places and times. Similarly, since $\mathfrak{E} = c\mathbf{E}/v_0B_0$, the electric field $\mathbf{E}(\mathbf{r}, t)$ must be increased by f. Since $\mathfrak{G}$ does not contain B_0, the $\mathbf{g}$ field is unchanged, as are the initial position $\mathbf{r}_0$ and direction $\mathbf{v}_0$.

Secondly, suppose v_0 is decreased by a factor $f < 1$, while keeping B_0 and L unchanged. This is not so simple as the first case, since the dimensionless time $\mathfrak{z}$ contains v_0. The requirement that $\mathfrak{B}(\mathfrak{R}, \mathfrak{z})$ be unchanged in the new experiment for a given $\mathfrak{z}$ means the time in which $\mathbf{B}$ varies must be increased by the factor $1/f$. Explicitly, $\mathbf{B}'(\mathbf{r}, t) = \mathbf{B}(\mathbf{r}, ft)$, where the prime means the field in the new experiment. For then $\mathfrak{B}'(\mathfrak{z}) = \mathbf{B}'(L\mathfrak{z}/v_0')/B_0'(L\mathfrak{z}/v_0') = \mathbf{B}(L\mathfrak{z}/v_0)/B_0(L\mathfrak{z}/v_0) = \mathfrak{B}(\mathfrak{z})$, so that $\mathfrak{B}'(\mathfrak{R}, \mathfrak{z}) = \mathfrak{B}(\mathfrak{R}, \mathfrak{z})$ as required. The condition that $\mathfrak{E}(\mathfrak{R}, \mathfrak{z})$ be unchanged means that $\mathfrak{E}$ must be reduced by f and the time in which it varies increased by $1/f$—that is, $\mathbf{E}'(\mathbf{r}, t) = f\mathbf{E}(\mathbf{r}, ft)$. The $\mathbf{g}$ field must be reduced by f^2 and the time scale increased, so that $\mathbf{g}'(\mathbf{r}, t) = f^2g(\mathbf{r}, ft)$. Again $\mathbf{r}_0$ and $\mathbf{v}_0$ are left unchanged.

Finally, suppose the size L of the system is increased by a factor $f > 1$, while

[1] In the absence of electric fields, β is constant and a relativistic particle follows the same trajectory as a nonrelativistic particle of the same velocity and same total mass. The work of the present paper therefore is applicable. Equations for the guiding center motion of a particle with relativistic energy in the presence of a small electric field have been given in Ref. *6*. Chandrasekhar and Vandervoort have studied the relativistic case in detail (private communication).

keeping B_0 and v_0 unchanged. In this case $\mathbf{r}_0$ must be increased by f so as to keep the initial $\mathfrak{R}$ unchanged. In addition, both the time and distance in which $\mathbf{E}$ and $\mathbf{B}$ change must be increased, so that $\mathbf{B}'(\mathbf{r}, t) = \mathbf{B}(\mathbf{r}/f, t/f)$ and $\mathbf{E}'(\mathbf{r}, t) = \mathbf{E}(\mathbf{r}/f, t/f)$. For then $\mathfrak{B}'(\mathfrak{R}, \mathfrak{J}) = \mathbf{B}'(L'\mathfrak{R}, L'\mathfrak{J}/v_0)/B_0'(L'\mathfrak{R}_1, L'\mathfrak{J}/v_0) = \mathbf{B}(L\mathfrak{R}, L\mathfrak{J}/v_0)/B_0(L\mathfrak{R}, L\mathfrak{J}/v_0) = \mathfrak{B}(\mathfrak{R}, \mathfrak{J})$ where $\mathfrak{R}_1 = \mathbf{r}_0/L = \mathbf{r}_0 f/L'$. Similarly, $\mathfrak{G}'(\mathfrak{R}, \mathfrak{J}) = \mathfrak{G}(\mathfrak{R}, \mathfrak{J})$. The $\mathbf{g}$ field must be changed both in magnitude and time scale, so that $g'(\mathbf{r}, t) = (1/f)\mathbf{g}(\mathbf{r}/f, t/f)$. The initial direction $\mathbf{v}_0$ is unchanged.

It is apparent that in all three cases the scaling of the various quantities decreases both the ratio of gyration radius to L, and the ratio of gyration period to the time scale in which the fields change, just as decreasing m/e does. Hence any one of the four parameters m/e, $1/B_0$, v_0, $1/L$ can be used as the parameter which is to be made smaller in order to make the guiding center equations derived below more closely represent the actual particle motion. In this paper $m/e \equiv \epsilon$ will be used, as Kruskal has done (*3*). The advantage of m/e over any one of the other three parameters (or a combination of them) is that the small quantity ϵ appears explicitly in the equation of motion of a particle without writing it in dimensionless form, whereas B_0, v_0, and L do not.

III. THE EQUATION OF MOTION OF THE GUIDING CENTER

To derive the equation of motion of the guiding center, let $\mathbf{r} = \mathbf{R} + \boldsymbol{\rho}$, where $\mathbf{r}$ is the instantaneous position of the particle, $\mathbf{R}$ is the position of the guiding center, and $\boldsymbol{\rho}$ is a vector from the guiding center to the particle (Fig. 1). The vector $\boldsymbol{\rho}$ can be given a precise definition as is done in Ref. (*5*) by the equation $\boldsymbol{\rho} = (mc/eB^2)\mathbf{B} \times (\mathbf{v} - c\mathbf{E} \times \mathbf{B}/B^2)$, where $\mathbf{E}$ and $\mathbf{B}$ are evaluated at $\mathbf{r}$. This combined with $\mathbf{r} = \mathbf{R} + \boldsymbol{\rho}$ gives a precise definition of $\mathbf{R}$. To lowest order in ϵ the fields can of course be evaluated at either $\mathbf{r}$ or $\mathbf{R}$, the difference being of order ϵ^2 in the equation for $\boldsymbol{\rho}$. Now substitute $\mathbf{r} = \mathbf{R} + \boldsymbol{\rho}$ into the nonrelativistic form of Eq. (1). Since the radius of gyration is proportional to ϵ, terms containing ρ^2 can be neglected compared to those in ρ.

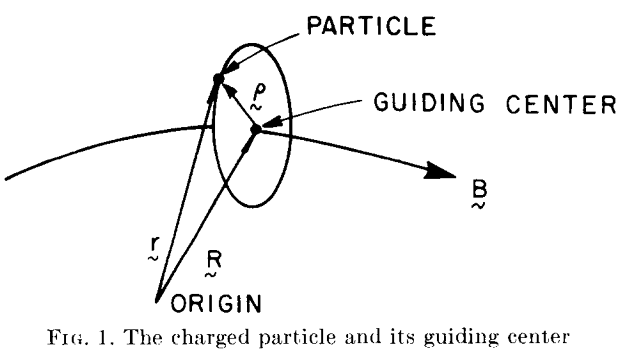

FIG. 1. The charged particle and its guiding center

The result of substituting $\mathbf{r} = \mathbf{R} + \boldsymbol{\rho}$ into the nonrelativistic form of Eq. (1) and expanding the fields in a Taylor series about $\mathbf{R}$ is

$$\ddot{\mathbf{R}} + \ddot{\boldsymbol{\rho}} = \mathbf{g} + (e/m)\{\mathbf{E}(\mathbf{R}) + \boldsymbol{\rho}\cdot\nabla\mathbf{E}(\mathbf{R}) + (1/c)(\dot{\mathbf{R}} + \dot{\boldsymbol{\rho}}) \times [\mathbf{B}(\mathbf{R}) + \boldsymbol{\rho}\cdot\nabla\mathbf{B}(\mathbf{R})]\} + o(\epsilon). \tag{4}$$

The term $(\dot{\boldsymbol{\rho}}/c) \times \boldsymbol{\rho}\cdot\nabla\mathbf{B}(\mathbf{R})$ must be retained in Eq. (4); as will become apparent shortly, this term is not of order ϵ^2 but is of order ϵ. Now define three orthogonal unit vectors: let $\mathbf{e}_1$ equal $\mathbf{B}/B$, let $\mathbf{e}_2$ be a unit vector directed towards the center of curvature of the line of force, and let $\mathbf{e}_3$ be $\mathbf{e}_1 \times \mathbf{e}_2$, a unit vector along the binormal. In order to correspond to the picture of the particle moving about a circle of radius ρ, let $\boldsymbol{\rho} = \rho(\mathbf{e}_2 \sin\theta + \mathbf{e}_3 \cos\theta)$, where $\theta = \int \omega\, dt$, ω being the gyro frequency $eB(\mathbf{R})/mc$. Then $\dot{\boldsymbol{\rho}} = \omega\rho(\mathbf{e}_2 \cos\theta - \mathbf{e}_3 \sin\theta) + (\rho\mathbf{e}_2)^{\cdot}\sin\theta + (\rho\mathbf{e}_3)^{\cdot}\cos\theta$. The first term contains $\omega\rho$ and is of zero order in ϵ, since $\omega \sim 1/\epsilon$ and $\rho \sim \epsilon$. The second and third terms contain ρ or $\dot{\rho}$ and are of order ϵ. The reason for retaining $\dot{\boldsymbol{\rho}} \times (\boldsymbol{\rho}\cdot\nabla)\mathbf{B}$ in Eq. (4) is now formally apparent, since it is of order ϵ, whereas a term such as $(\boldsymbol{\rho}\cdot\nabla)^2\mathbf{E}$ in the Taylor expansion is of order ϵ^2. A second differentiation gives $\ddot{\boldsymbol{\rho}} = -[\omega^2\rho(\mathbf{e}_2 \sin\theta + \mathbf{e}_3 \cos\theta)] + \dot{\omega}\rho[\mathbf{e}_2 \cos\theta - \mathbf{e}_3 \sin\theta] + 2\omega[(\rho\mathbf{e}_2)^{\cdot}\cos\theta - (\rho\mathbf{e}_3)^{\cdot}\sin\theta] + [(\rho\mathbf{e}_2)^{\cdot\cdot}\sin\theta + (\rho\mathbf{e}_3)^{\cdot\cdot}\cos\theta]$, the four terms being of order $1/\epsilon$, 1, 1, and ϵ, respectively. These expressions for $\boldsymbol{\rho}$, $\dot{\boldsymbol{\rho}}$, $\ddot{\boldsymbol{\rho}}$ and are now substituted into Eq. (4) and the resulting equation time-averaged over a gyration period, by taking $\int_0^{2\pi}(\cdots)\, d\theta$ and considering coefficients, such as $(\rho\mathbf{e}_2)^{\cdot}$, to be constants. Then $\langle\boldsymbol{\rho}\rangle = \langle\dot{\boldsymbol{\rho}}\rangle = \langle\ddot{\boldsymbol{\rho}}\rangle = 0$, where the angular brackets denote the average. The result of time-averaging Eq. (4) is

$$\ddot{\mathbf{R}} = \mathbf{g}(\mathbf{R}) + \frac{e}{m}\left[\mathbf{E}(\mathbf{R}) + \frac{\dot{\mathbf{R}}}{c} \times \mathbf{B}(\mathbf{R})\right] + \frac{e}{mc}\frac{\rho^2\omega}{2}[\mathbf{e}_2 \times (\mathbf{e}_3\cdot\nabla)\mathbf{B} - \mathbf{e}_3 \times (\mathbf{e}_2\cdot\nabla)\mathbf{B}] + o(\epsilon), \tag{5}$$

since

$$\langle\dot{\boldsymbol{\rho}} \times (\boldsymbol{\rho}\cdot\nabla)\mathbf{B}\rangle = (\rho^2\omega/2)[\mathbf{e}_2 \times (\mathbf{e}_3\cdot\nabla)\mathbf{B} - \mathbf{e}_3 \times (\mathbf{e}_2\cdot\nabla)\mathbf{B}]. \tag{6}$$

The coefficient $\rho^2\omega/2$ is an approximate invariant of the motion and is Mc/e, where M is the well-known magnetic moment. That M is an adiabatic invariant of the particle motion has been demonstrated in Ref. *3* and in numerous other places. Equation (5) has essentially been derived by Kruskal.[2]

[2] The derivation of the guiding center equation in the present paper is intuitive rather than mathematically rigorous. The work of Kruskal (*3*) and of Berkowitz and Gardner (*4*) constitute the rigorous justification for the averaging process used to get Eq. (5). Kruskal derives equations for the $\mathbf{R}_n$ appearing in a series of the form

$$\mathbf{r} = \sum_{-\infty}^{\infty} \epsilon^{|n|}\mathbf{R}_n(t) \exp\left(in \int \omega(\mathbf{R}_0)\, dt\right) \equiv \mathbf{R}_0 + \sum_{1}^{\infty} \epsilon^n\left(\boldsymbol{\alpha}_n \sin n \int \omega\, dt + \boldsymbol{\beta}_n \cos n \int \omega\, dt\right)$$

The right-hand side of Eq. (6) can be simplified as follows:

$$\mathbf{e}_2 \times (\mathbf{e}_3\cdot\nabla)\mathbf{B} = (\mathbf{e}_3 \times \mathbf{e}_1) \times (\mathbf{e}_3\cdot\nabla)\mathbf{B} = \mathbf{e}_1[\mathbf{e}_3\cdot(\mathbf{e}_3\cdot\nabla)\mathbf{B}] - \mathbf{e}_3[\mathbf{e}_1\cdot(\mathbf{e}_3\cdot\nabla)\mathbf{B}]. \quad (7)$$

Now

$$\mathbf{e}_1\cdot(\mathbf{e}_3\cdot\nabla)\mathbf{B} = \mathbf{e}_1\cdot(\mathbf{e}_3\cdot\nabla)(\mathbf{e}_1 B) = (B/2)(\mathbf{e}_3\cdot\nabla)\mathbf{e}_1^2 + \mathbf{e}_3\cdot\nabla B = \mathbf{e}_3\cdot\nabla B, \quad (8)$$

since $\mathbf{e}_1^2 = 1$. Therefore Eq. (7) becomes

$$\mathbf{e}_2 \times (\mathbf{e}_3\cdot\nabla)\mathbf{B} = \mathbf{e}_1[\mathbf{e}_3\cdot(\mathbf{e}_3\cdot\nabla)\mathbf{B}] - \mathbf{e}_3(\mathbf{e}_3\cdot\nabla)B. \quad (9)$$

Similarly

$$\mathbf{e}_3 \times (\mathbf{e}_2\cdot\nabla)\mathbf{B} = -\mathbf{e}_1[\mathbf{e}_2\cdot(\mathbf{e}_2\cdot\nabla)\mathbf{B}] + \mathbf{e}_2(\mathbf{e}_2\cdot\nabla)B. \quad (10)$$

The fact that $\nabla\cdot\mathbf{B} = 0$ must now be used. The operator ∇ can be expressed as

$$\mathbf{e}_1(\mathbf{e}_1\cdot\nabla) + \mathbf{e}_2(\mathbf{e}_2\cdot\nabla) + \mathbf{e}_3(\mathbf{e}_3\cdot\nabla),$$

so that

$$\nabla\cdot\mathbf{B} = \mathbf{e}_1\cdot(\mathbf{e}_1\cdot\nabla)\mathbf{B} + \mathbf{e}_2\cdot(\mathbf{e}_2\cdot\nabla)\mathbf{B} + \mathbf{e}_3\cdot(\mathbf{e}_3\cdot\nabla)\mathbf{B} = 0.$$

But

$$\mathbf{e}_1\cdot(\mathbf{e}_1\cdot\nabla)\mathbf{B} = \mathbf{e}_1\cdot\partial\mathbf{B}/\partial s = \partial B/\partial s,$$

where s is distance along the line of force. Therefore by subtracting (10) from (9) and using $\nabla\cdot\mathbf{B} = 0$, one obtains

$$\begin{aligned}\mathbf{e}_2 \times (\mathbf{e}_3\cdot\nabla)\mathbf{B} &- \mathbf{e}_3 \times (\mathbf{e}_2\cdot\nabla)\mathbf{B} \\ &= -\mathbf{e}_1(\partial B/\partial s) - \mathbf{e}_2(\mathbf{e}_2\cdot\nabla)B - \mathbf{e}_3(\mathbf{e}_3\cdot\nabla)B = -\nabla B.\end{aligned} \quad (11)$$

The time average of Eq. (4) then is

$$\boxed{\ddot{\mathbf{R}} = \mathbf{g}(\mathbf{R}) + (e/m)[\mathbf{E}(\mathbf{R}) + (\dot{\mathbf{R}}/c) \times \mathbf{B}(\mathbf{R})] - (M/m)\nabla B(\mathbf{R}) + o(\epsilon),} \quad (12)$$

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by the formal process of equating coefficients of equal powers of exp $(i \int \omega\, dt)$. The equation for $\mathbf{R}_0$ is just Eq.(5). It is not immediately obvious that Kruskal's procedure of equating coefficients is justified, for the coefficients $\mathbf{R}_n$ are functions of time so that this is not simply a Fourier series. However, Berkowitz and Gardner provide the justification by proving that this series is indeed the asymptotic expansion of $\mathbf{r}$ for small ϵ. The series is actually a generalization of a WKB series to the case of a nonlinear differential equation. The guiding center $\mathbf{R}$, as defined at the start of Section III, differs from $\mathbf{R}_0$ by $o(\epsilon^2)$. $\mathbf{R} = \mathbf{R}_0 + o(\epsilon^2)$. This difference is of no consequence here, since we consider only effects that are first order in the radius of gyration.

The general asymptotic theory of systems of ordinary differential equations with nearly periodic solutions has been studied by Bogoliubov and Zubarev (7) and by Kruskal (7) in a book which is a collection of lectures given at the summer school of theoretical physics at Les Houches in the summer of 1959. In each work the general theory has been illustrated by the equation of motion of a charged particle. Bogoliubov obtains the longitudinal equation of motion (20) and the drift velocity (17) for the case where $\mathbf{u}_E$ is $o(\epsilon)$. Kruskal's emphasis is on the adiabatic invariants.

with an initial velocity $\dot{\mathbf{R}}(0)$ equal to $[\mathbf{e}_1\mathbf{e}_1\cdot\mathbf{v} + (c\mathbf{E} \times \mathbf{B}/B^2)]|_{t=0} + o(\epsilon)$. Equation (12) is the basic differential equation for the guiding center motion. Hellwig (*2*) has given a similar but more lengthy derivation of it. It is the same as the equation of motion of a particle in a magnetic field $\mathbf{B}$ and an equivalent electric field $\mathbf{E} - (M/e)\nabla B$, and therefore for numerical integration is more complicated than was the original equation of motion (1). If a numerical solution of (12) were performed, it would be found that the guiding center $\mathbf{R}$ travels in roughly a helix about the field line, just as the particle does. However, it can be shown that the radius of this helix is one order of ϵ smaller than the radius of gyration of the particle, as would be expected for the guiding center. This small amplitude oscillation of the guiding center is to be ignored, since it is of order ϵ^2 and of no significance in a first-order theory. Furthermore in the preceding analysis other terms of this order have been neglected. In the next section Eq. (12) will be solved by iteration to obtain the equations for the guiding center motion parallel and perpendicular to $\mathbf{B}$. These coupled equations do not show the rapid guiding center spiralling that Eq. (12) does.

IV. THE DRIFT VELOCITY AND THE LONGITUDINAL MOTION

The differential equation for the guiding center motion can be separated into components parallel and perpendicular to $\mathbf{B}$. Crossing Eq. (12) on the right with $\mathbf{e}_1(\mathbf{R})$ gives the perpendicular component of the vector equation as

$$\dot{\mathbf{R}} - \mathbf{e}_1\cdot\dot{\mathbf{R}} = \dot{\mathbf{R}}_\perp = \frac{c\mathbf{E} \times \mathbf{e}_1}{B} + \frac{Mc}{e}\frac{\mathbf{e}_1 \times \nabla B}{B} + \frac{mc}{e}\frac{(\mathbf{g} - \ddot{\mathbf{R}}) \times \mathbf{e}_1}{B} + o(\epsilon^2), \quad (13)$$

where $\dot{\mathbf{R}}_\perp$ is the component of $\dot{\mathbf{R}}$ perpendicular to $\mathbf{e}_1(\mathbf{R})$. It is called the drift velocity. The first term is the usual "$\mathbf{E} \times \mathbf{B}$" drift. The second term is the "gradient B" drift, and the third is the "acceleration drift". By dotting Eq. (12) with $\mathbf{e}_1(\mathbf{R})$ one obtains the scalar parallel component as

$$\frac{m}{e}\ddot{\mathbf{R}}\cdot\mathbf{e}_1 = \frac{m}{e}\mathbf{g}\cdot\mathbf{e}_1 + \mathbf{E}\cdot\mathbf{e}_1 - \frac{M}{e}\frac{\partial B}{\partial s} + o(\epsilon^2). \quad (14)$$

In Eq. (13) the guiding center acceleration $\ddot{\mathbf{R}}$ is needed to calculate the drift velocity; but because the term in which it occurs already contains ϵ as a coefficient, $\ddot{\mathbf{R}}$ is needed only to zero order in ϵ. It is assumed that $\ddot{\mathbf{R}}$ is not of negative order, such as $\sim 1/\epsilon$. If it were of negative order, the fields would change by a large amount in a gyration period when ϵ is small, and the guiding center picture would not be valid. Neither would M be an adiabatic invariant (see Ref. *3*).

The acceleration $\ddot{\mathbf{R}} = d\dot{\mathbf{R}}/dt = (d/dt)(\dot{\mathbf{R}}_\perp + \mathbf{e}_1\dot{\mathbf{R}}\cdot\mathbf{e}_1)$, and $d\dot{\mathbf{R}}_\perp/dt$ can be obtained to zero order in ϵ from Eq. (13) as

$$d\dot{\mathbf{R}}_\perp/dt = (d/dt)(c\mathbf{E} \times \mathbf{e}_1/B) + o(\epsilon).$$

Only the first term in the drift is needed, since the third term is $\sim\epsilon$ and the second term contains $M/e = m(\rho\omega)^2/2eB \sim \epsilon$. If the perpendicular electric field happens to be of order ϵ instead of zero order, the retention of $c\mathbf{E} \times \mathbf{e}_1/B \equiv \mathbf{u}_E$ would be unnecessary in the calculation of $\ddot{\mathbf{R}}$. The acceleration then is

$$\ddot{\mathbf{R}} = (d/dt)\dot{\mathbf{R}} = (d/dt)(v_\parallel \mathbf{e}_1 + \mathbf{u}_E) + o(\epsilon) \tag{15}$$

$$\begin{aligned}
\ddot{\mathbf{R}} - o(\epsilon) &= \mathbf{e}_1 \frac{dv_\parallel}{dt} + v_\parallel \frac{d\mathbf{e}_1}{dt} + \frac{d\mathbf{u}_E}{dt} \\
&= \mathbf{e}_1 \frac{dv_\parallel}{dt} + v_\parallel \left[\frac{\partial \mathbf{e}_1}{\partial t} + (\mathbf{e}_1 v_\parallel + \mathbf{u}_E) \cdot \nabla \mathbf{e}_1 \right] \\
&\qquad + \left[\frac{\partial \mathbf{u}_E}{\partial t} + (\mathbf{e}_1 v_\parallel + \mathbf{u}_E) \cdot \nabla \mathbf{u}_E \right] \qquad (16) \\
&= \mathbf{e}_1 \frac{dv_\parallel}{dt} + v_\parallel \frac{\partial \mathbf{e}_1}{\partial t} + v_\parallel^2 \frac{\partial \mathbf{e}_1}{\partial s} \\
&\qquad + v_\parallel \mathbf{u}_E \cdot \nabla \mathbf{e}_1 + \frac{\partial \mathbf{u}_E}{\partial t} + v_\parallel \frac{\partial \mathbf{u}_E}{\partial s} + \mathbf{u}_E \cdot \nabla \mathbf{u}_E ,
\end{aligned}$$

where $v_\parallel$ means $\dot{\mathbf{R}} \cdot \mathbf{e}_1(\mathbf{R})$. Other "parallel" velocities can be defined, such as $\mathbf{v} \cdot \mathbf{e}_1(\mathbf{r})$ or $\mathbf{v} \cdot \mathbf{e}_1(\mathbf{R})$, but in this paper $v_\parallel$ always stands for $\dot{\mathbf{R}} \cdot \mathbf{e}_1(\mathbf{R})$. The first term is the tangential acceleration, the third is the centripetal acceleration, the second occurs in nonstatic fields, where the direction of the line of force changes with time, while the last four terms occur in the presence of a zero-order electric field. Again it should be stated that the presence of a "zero-order electric field" means that in a series of experiments in which m/e is successively reduced, the electric field is held constant. The electric field is of order ϵ if it is reduced in proportion to m/e. Whether the $\partial \mathbf{e}_1/\partial t$ term need be retained or not depends on how the time in which fields vary is to be scaled in the series of experiments. If the time scale is held constant, then $\partial/\partial t$ is of zero order and $\partial \mathbf{e}_1/\partial t$ contributes a first-order drift. If the time scale is increased in proportion to $1/\epsilon$, then $\partial/\partial t$ is of order ϵ and $\partial \mathbf{e}_1/\partial t$ is not needed.

With expression (16) for $\ddot{\mathbf{R}}$, Eq. (13) for the drift becomes

$$\begin{aligned}
\dot{\mathbf{R}}_\perp = \frac{\mathbf{e}_1}{B} \times \Bigg\{ -c\mathbf{E} + \frac{Mc}{e} \nabla B + \frac{mc}{e} \bigg[-\mathbf{g} + v_\parallel \frac{\partial \mathbf{e}_1}{\partial t} + v_\parallel^2 \frac{\partial \mathbf{e}_1}{\partial s} + v_\parallel \mathbf{u}_E \cdot \nabla \mathbf{e}_1 \\
+ \frac{\partial}{\partial t} \mathbf{u}_E + v_\parallel \frac{\partial}{\partial s} \mathbf{u}_E + \mathbf{u}_E \cdot \nabla \mathbf{u}_E \bigg] \Bigg\} + o(\epsilon^2),
\end{aligned} \tag{17}$$

where $\mathbf{u}_E = c\mathbf{E} \times \mathbf{e}_1/B$. These drift terms will be illustrated by examples in Section V.

The longitudinal Eq. (14) shows $E_\parallel = \mathbf{E} \cdot \mathbf{e}_1$ must be of order ϵ if $\ddot{\mathbf{R}}$ is to be of non-negative order. Thus in contrast to $\mathbf{E}_\perp$, which may be of zero order, $E_\parallel$ must be of order ϵ. If this were not so, the parallel acceleration would be $\sim 1/\epsilon$.

Equation (14) can be put in a form more useful for obtaining an energy integral by rewriting $\ddot{\mathbf{R}}\cdot\mathbf{e}_1$ as

$$\ddot{\mathbf{R}}\cdot\mathbf{e}_1 = (d/dt)(\dot{\mathbf{R}}\cdot\mathbf{e}_1) - \dot{\mathbf{R}}\cdot\dot{\mathbf{e}}_1 = (dv_{\parallel}/dt) - \dot{\mathbf{R}}\cdot\dot{\mathbf{e}}_1 \tag{18}$$

and noting that

$$\begin{aligned}\dot{\mathbf{R}}\cdot\dot{\mathbf{e}}_1 &= (\mathbf{e}_1 v_{\parallel} + \mathbf{u}_E + o(\epsilon))\cdot\dot{\mathbf{e}}_1 = \mathbf{u}_E\cdot\dot{\mathbf{e}}_1 + o(\epsilon)\\ &= \mathbf{u}_E\cdot[(\partial\mathbf{e}_1/\partial t) + (\mathbf{e}_1 v_{\parallel} + \mathbf{u}_E)\cdot\nabla\mathbf{e}_1] + o(\epsilon).\end{aligned} \tag{19}$$

In Eqs. (18) and (19) we consider only the contribution of the zero order motion to d/dt. This is all that is required, since $\ddot{\mathbf{R}}\cdot\mathbf{e}_1$ has ϵ for a coefficient in Eq. (14). The longitudinal Eq. (14) then becomes

$$\frac{m}{e}\frac{dv_{\parallel}}{dt} = \frac{m}{c}g_{\parallel} + E_{\parallel} - \frac{M}{e}\frac{\partial B}{\partial s} + \frac{m}{e}\mathbf{u}_E\cdot\left(\frac{\partial\mathbf{e}_1}{\partial t} + v_{\parallel}\frac{\partial\mathbf{e}_1}{\partial s} + \mathbf{u}_E\cdot\nabla\mathbf{e}_1\right) + o(\epsilon^2). \tag{20}$$

Equations (17) and (20) are equivalent to the original differential Eq. (12).[3]

Let us now introduce a true curvilinear coordinate system (α, β, s) such that $\alpha(\mathbf{r}, t)$ and $\beta(\mathbf{r}, t)$ are two parameters specifying a line of force and therefore constant on it; s is distance along the line as previously. For a divergence-free field such as $\mathbf{B}$, α, and β can be chosen so that the vector potential is $\mathbf{A} = \alpha\nabla\beta$. Then

$$E_{\parallel} = \mathbf{e}_1\cdot\left(-\frac{1}{c}\frac{\partial\mathbf{A}}{\partial t} - \nabla\varphi\right) = -\frac{\partial}{\partial s}(\psi + \varphi), \tag{21}$$

where $\psi = (\alpha/c)(\partial\beta/\partial t)$. Now the rate of change of $\varphi + \psi$ due to zero order motion and time-dependent fields is

$$\begin{aligned}\frac{d(\psi+\varphi)}{dt} &= (\mathbf{e}_1 v_{\parallel} + \mathbf{u}_E)\cdot\nabla(\psi+\varphi) + \frac{\partial(\psi+\varphi)}{\partial t}\\ &= v_{\parallel}\frac{\partial(\psi+\varphi)}{\partial s} + \mathbf{u}_E\cdot\nabla(\psi+\varphi) + \frac{\partial(\psi+\varphi)}{\partial t}\end{aligned} \tag{22}$$

[3] The experimental physicist may at this point ask how he is to know whether the electric field drift $\mathbf{u}_E$ is $o(1)$ or $o(\epsilon)$ in a *given* piece of experimental equipment. Or differently asked, at how many volts per meter electric field does his $\mathbf{u}_E$ become $o(1)$ instead of $o(\epsilon)$, thus requiring him to retain terms with $\mathbf{u}_E$ in Eqs. (17) and (20). The answer is that it never in principle is wrong to keep these terms. For the given experiment they may be much less than the other terms, in which case they could have been omitted. The guiding center equations are merely guides to what the particle may be expected to do. The equations are derived from an asymptotic series, and therefore become better predictions of the actual particle motion as the expansion parameter ϵ is decreased. How good predictions they are in any particular experiment could be determined by comparison with a detailed numerical solution of the particle orbit, or less accurately by looking at the magnitude of next higher order terms.

so that

$$v_\parallel E_\parallel = -v_\parallel \frac{\partial(\psi + \varphi)}{\partial s}$$
$$= -\frac{d(\psi + \varphi)}{dt} + \mathbf{u}_E \cdot \nabla(\psi + \varphi) + \frac{\partial(\psi + \varphi)}{\partial t} + o(\epsilon^2). \tag{23}$$

Similarly

$$v_\parallel \frac{\partial B}{\partial s} = \frac{dB}{dt} - \mathbf{u}_E \cdot \nabla B - \frac{\partial B}{\partial t}. \tag{24}$$

When $E_\parallel$ and $\partial B/\partial s$ are eliminated from Eq. (20) by use of Eqs. (23) and (24), the result is

$$\frac{d}{dt}\left(\frac{m}{2e} v_\parallel^2 + \frac{M}{e} B + \psi + \varphi\right) = \frac{m}{e} g_\parallel v_\parallel + \mathbf{u}_E \cdot \left[\nabla \left(\frac{M}{e} B + \psi + \varphi\right) + \frac{m}{e} v_\parallel \left(\frac{\partial \mathbf{e}_1}{\partial t} + v_\parallel \frac{\partial \mathbf{e}_1}{\partial s} + \mathbf{u}_E \cdot \nabla \mathbf{e}_1\right)\right] + \frac{\partial}{\partial t}\left(\frac{M}{e} B + \psi + \varphi\right) + o(\epsilon^2), \tag{25}$$

a form which will be useful in the applications of the next section. If the **g** field can be derived from a potential, this potential will appear added to

$$(M/e)B + \psi + \varphi$$

and the $(m/e)g_\parallel$ term will be absent in Eq. (25).

V. APPLICATIONS

A. Dielectric Constant of a Plasma

Suppose there is a uniform magnetic field out of the page (Fig. 2) and a

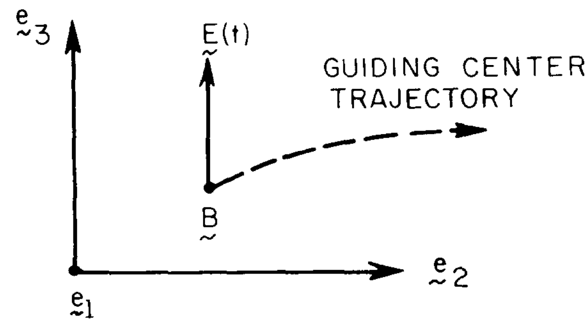

Fig. 2. Polarization of a plasma is caused by the component of guiding center drift parallel to the electric field.

spatially uniform but time-dependent electric field at right angles. From Eq. (17), if $\partial/\partial t \sim 1$ and $\mathbf{E}_\perp \sim 1$,

$$\begin{aligned}\dot{\mathbf{R}}_\perp &= \frac{c\mathbf{E} \times \mathbf{e}_1}{B} + \frac{mc}{eB}\mathbf{e}_1 \times \frac{\partial}{\partial t}\left(\frac{c\mathbf{E} \times \mathbf{e}_1}{B}\right) \\ &= \frac{c\mathbf{E} \times \mathbf{e}_1}{B} - \frac{mc^2}{eB^2}\mathbf{e}_1 \times \left(\mathbf{e}_1 \times \frac{\partial \mathbf{E}}{\partial t}\right) = \frac{c\mathbf{E} \times \mathbf{e}_1}{B} + \frac{mc^2}{eB^2}\frac{\partial \mathbf{E}}{\partial t}.\end{aligned} \tag{26}$$

Because the average position of a particle is its guiding center, the polarization of the medium can be obtained from the displacement of the guiding center in the direction of the **E** field—that is,

$$\int_0^t dt\, \dot{\mathbf{R}}_\perp \cdot \mathbf{e}_3 = \frac{mc^2}{eB^2}\int_0^t \mathbf{e}_3 \cdot \frac{\partial \mathbf{E}}{\partial t}\, dt = \frac{mc^2}{eB^2}\Delta E, \tag{27}$$

where ΔE is the change in the magnitude of the field from time zero to t. The polarizability is then, $\chi = nmc^2/B^2$, where n is the ion density. The electrons have a negligible polarizability, since the electron mass is much less than the ion mass. The dielectric constant κ is

$$\kappa = 1 + 4\pi\chi = 1 + (4\pi nmc^2/B^2). \tag{28}$$

Another way to derive χ is from energy considerations. The kinetic energy for the increase in the first term of Eq. (26), the $\mathbf{E} \times \mathbf{B}$ drift, must be supplied by motion in the direction of the **E** field. Then

$$(m/2)\Delta(cE/B)^2 = eE\int_0^t dt\, \dot{\mathbf{R}}_\perp \cdot \mathbf{e}_3, \quad \text{or} \quad \int_0^t dt\, \dot{\mathbf{R}}_\perp \cdot \mathbf{e}_3 = (mc^2/eB^2)\Delta E$$

as in eq. (27).

B. Magnetic Mirror Reflection when Electric Fields are Small

Consider a magnetic mirror geometry (Fig. 3) in which the fields are static or slowly varying ($\partial/\partial t \sim \epsilon$) and in which $E_\perp$ is of order ϵ, so that $\mathbf{u}_E \sim \epsilon$. The guiding center velocity $\dot{\mathbf{R}}$ then is to lowest order $\mathbf{e}_1 v_\parallel$ along a line of force. In

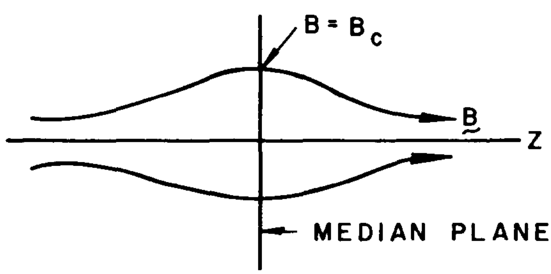

Fig. 3. Magnetic mirror machine

Eq. (25), the entire right-hand side then is of $o(\epsilon^2)$. For φ must be $\sim\epsilon$ if both $\mathbf{E}_\perp$ and $E_\parallel \sim \epsilon$. Also $\psi = (\alpha/c)(\partial\beta/\partial t) \sim \partial/\partial t \sim \epsilon$. Thus Eq. (25) becomes (whether there is rotational symmetry or not)

$$(m/2)v_\parallel^2 + MB + e(\psi + \varphi) \quad \text{equals a constant of the zero-order motion along the line of force.}^4 \tag{29}$$

The particle reflects when $v_\parallel = 0$, which occurs at a field B_T defined by $MB_T + e(\psi_T + \varphi_T) =$ the constant of the motion.

If the magnetic field is static, $\psi = 0$, and Eq. (29) is merely the conservation of total energy $H = (mv^2/2) + e\varphi$. If there are no electric fields so that $\psi = \varphi = 0$, the familiar result for a mirror machine is obtained: $B_T = H/M = B_c/\sin^2 \delta$, where δ is the angle between the velocity vector and the field line at the median plane, and B_c is the field there.

C. Limiting Time of Oscillation between Mirror Points

For small amplitude oscillations about the median plane of Fig. 3, $B \cong B_c + (s^2/2)B_c''$, where $s = 0$ at the median plane and B_c'' is $(d^2B/ds^2)s = 0$. Equation (20) is $d^2s/dt^2 = -(M/m)B_c''s$, which is the equation of motion of a harmonic oscillator with period $2\pi(m/MB_c'')^{1/2} = (2\pi/v)(2B_c/B_c'')^{1/2}$.

D. Magnetic Mirror Reflection when the Electric field is Large

In this section the reflection properties of a magnetic mirror will be determined for the case where $\mathbf{E}_\perp$ is of zero order. If $\mathbf{E}_\perp$ is of zero order in ϵ, the right-hand side of Eq. (25) stands, with no apparent simplification possible in general. However, in at least one special case the equation can be written in the form of an energy integral. This is the case of a static magnetic field with rotational symmetry (such as a mirror machine), and a static $\mathbf{E}$, where $\mathbf{E}_\perp$ has no azimuthal component and $E_\parallel = 0$ (Fig. 4). Such a mirror machine has been named Ixion, and is discussed by Longmire *et al.* (*8*). Wilcox (*9*) has reviewed experimental results obtained with Ixion and similar machines. The zero-order drift $\mathbf{u}_E$ is in the azimuthal direction; the component parallel to $\mathbf{B}$ of the resulting radial centrifugal force mu_E^2/r has the desirable property of making it more difficult for the particle to escape at the ends. The effect is just that which would be observed if a bead were placed on a smooth wire bent in the shape of the line of force, and the wire then rotated about the z axis. This analogy will become apparent in the following analysis, which is quite different in method from that of Ref. *8*, but leads to the same results.

Under the specified restriction on the $\mathbf{E}$ and $\mathbf{B}$ fields, all terms on the right side of Eq. (25) vanish except the one containing $\mathbf{u}_E \cdot (\mathbf{u}_E \cdot \nabla)\mathbf{e}_1$, which in this special case equals $(cE/B)^2\mathbf{e}_3 \cdot (\mathbf{e}_3 \cdot \nabla)\mathbf{e}_1$. Because $\mathbf{e}_3 \cdot \mathbf{e}_1 = 0$, the factor $\mathbf{e}_3 \cdot (\mathbf{e}_3 \cdot \nabla)\mathbf{e}_1$

[4] The relativistic form of Eq. (29) is shown in Ref. *6* to be $(P^2c^2 + m_0^2c^4)^{1/2} + e(\psi + \varphi) =$ constant of the zero-order motion.

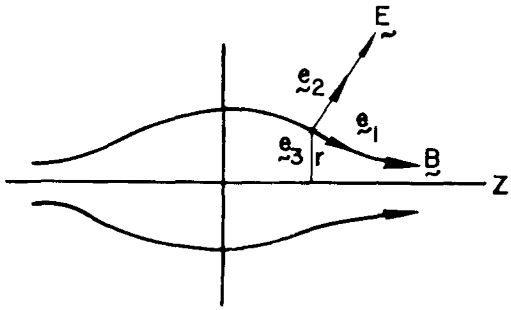

FIG. 4. Mirror machine with large electric field

equals $-\mathbf{e}_1\cdot(\mathbf{e}_3\cdot\nabla)\mathbf{e}_3$. But $(\mathbf{e}_3\cdot\nabla)\mathbf{e}_3 = -\mathbf{e}_r/r$, where $\mathbf{e}_r$ is a unit vector in the radial direction. Therefore $-\mathbf{e}_1\cdot(\mathbf{e}_3\cdot\nabla)\mathbf{e}_3 = \mathbf{e}_1\cdot\mathbf{e}_r/r$.

In order to integrate $(cE/B)^2(\mathbf{e}_1\cdot\mathbf{e}_r/r)$ over the zero-order motion on a flux surface (defined as the surface formed by revolving a line of force about z), the variation of cE/B and $\mathbf{e}_1\cdot\mathbf{e}_r/r$ with longitudinal position must be known. The following is a proof that cE/rB is independent of position on a flux surface. Let $\Psi(r, z)$ be the stream function (*10*) for the magnetic field; the stream function has the property that Ψ = constant is the equation of a line of force and that $B_z = (1/r)\partial\Psi/\partial r$ and $B_r = -(1/r)\partial\Psi/\partial z$. Since **E** is perpendicular to **B**, flux surfaces are also equipotentials and φ is therefore a function of Ψ. The components of electric field are

$$E_r = -\partial\varphi/\partial r = -(d\varphi/d\Psi)\partial\Psi/\partial r \text{ and } E_z = -(d\varphi/d\Psi)\partial\Psi/\partial z.$$

Thus $E = [(\partial\Psi/\partial r)^2 + (\partial\Psi/\partial z)^2]^{1/2}\, d\varphi/d\Psi = rB\,(d\varphi/d\Psi)$ and $cE/rB = c\,d\varphi/d\Psi$, which is constant on a flux surface. The quantity cE/rB is the angular velocity of the $\mathbf{u}_E$ drift about z and will be denoted by Ω. Therefore the term containing $\mathbf{u}_E\cdot(\mathbf{u}_E\cdot\nabla)\mathbf{e}_1$ in Eq. (25) is $(m/e)v_\parallel\Omega^2 r\mathbf{e}_1\cdot\mathbf{e}_r$, which equals $(m/e)\,(d/dt)\,(\Omega^2r^2/2)$, because

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}(\Omega^2r^2) &= \frac{\Omega^2}{2}\frac{dr^2}{dt} = \Omega^2 r\frac{dr}{dt} = \Omega^2 r(\mathbf{e}_1 v_\parallel + \mathbf{u}_E)\cdot\nabla r \\ &= \Omega^2 v_\parallel r\frac{\partial r}{\partial s} = \Omega^2 v_\parallel r(\mathbf{e}_1\cdot\mathbf{e}_r).\end{aligned} \tag{30}$$

The $d\varphi/dt$ term on the left side of Eq. (25) vanishes, since the zero-order motion is on an equipotential surface. The integral of Eq. (25) is then

$$mv_\parallel^2/2 + MB - m\Omega^2r^2/2 \quad \text{equals a constant of the zero-order motion on the flux surface.} \tag{31}$$

Equation (31) has been derived in a different fashion by Chandrasekhar *et al.* (*11*), and in yet a third way in Ref. (*8*).

If the subscript c designates quantities at the median plane of Fig. 4 and e at the mirror (i.e., at the location of maximum magnetic field on the flux surface), Eq. (31) becomes

$$v_{\parallel e}^2 = v_{\parallel c}^2 + (2MB_c/m)(1 - B_e/B_c) - \Omega^2 r_c^2(1 - r_e^2/r_c^2). \qquad (32)$$

In Ref. *3* Kruskal shows that if $\mathbf{u}_E \neq 0$, MB equals $mv_\perp^2/2B$, where $v_\perp$ is the perpendicular velocity in the frame of reference moving at velocity $\mathbf{u}_E$. Also B_e/B_c equals the mirror ratio on the flux surface. Equation (32) therefore says that $v_{\parallel e}^2 \leqq 0$—i.e., the particle is contained, if

$$v_{\parallel c}^2 \leqq v_{\perp c}^2[(B_e/B_c) - 1] + u_{Ec}^2(1 - r_e^2/r_c^2). \qquad (33)$$

If in addition the magnetic field is assumed to be approximately independent of radius both in the median plane and at the mirror, then by conservation of flux $r_e^2/r_c^2 = B_c/B_e$ in Eq. (33). However this assumption is not necessary for the validity of the adiabatic theory and Eq. (33).

If in Eq. (32) M is set equal to zero, the change in parallel kinetic energy between the median plane and the mirror is $(m/2)\Omega^2(r_c^2 - r_e^2)$, which is just the work done against the centrifugal force. Thus when $M = 0$, the problem is that of the bead sliding on the wire described previously.

Terms containing $\mathbf{u}_E$ in the drift Eq. (17) give a small (order ϵ) motion in or normal to a flux surface, the zero-order velocity being $\dot{\mathbf{R}} = \mathbf{e}_1 v_\parallel + \mathbf{u}_E$ in the surface. When crossed with $\mathbf{e}_1/B$, the third term in the square brackets is the usual drift due to line curvature and is in the azimuthal direction. If $\mathbf{E}$ is outward as in Fig. 4, the fourth term is

$$-v_\parallel \frac{cE}{B}\mathbf{e}_3\cdot\nabla\mathbf{e}_1 = -v_\parallel\Omega\frac{\partial\mathbf{e}_1}{\partial\theta} = -v_\parallel\Omega\frac{\partial}{\partial\theta}(\mathbf{e}_r\mathbf{e}_r\cdot\mathbf{e}_1 + \mathbf{e}_z\mathbf{e}_z\cdot\mathbf{e}_1) = -v_\parallel\Omega\mathbf{e}_3(\mathbf{e}_r\cdot\mathbf{e}_1),$$

where θ is the azimuthal angle in cylindrical coordinates and $\mathbf{e}_z$ is a unit vector in the z direction. When crossed with $\mathbf{e}_1$, this fourth term gives a drift normal to the flux surface. The sixth term in brackets is

$$v_\parallel(\partial\mathbf{u}_E/\partial s) = -v_\parallel(\partial/\partial s)(\Omega r\mathbf{e}_3) = -v_\parallel\Omega\mathbf{e}_3(\partial r/\partial s) = -v_\parallel\Omega\mathbf{e}_3(\mathbf{e}_r\cdot\mathbf{e}_1),$$

hence is the same in this geometry as the fourth term. The last term in the square brackets is

$$\Omega r\mathbf{e}_3\cdot\nabla(\Omega r\mathbf{e}_3) = \Omega^2 r^2(\mathbf{e}_3\cdot\nabla)\mathbf{e}_3 = -\Omega^2 r\mathbf{e}_r.$$

When crossed with $\mathbf{e}_1$ this last term gives another order ϵ drift in the surface, in addition to the ∇B and line curvature drifts.

Because of the two order ϵ drift terms perpendicular to the flux surface, there is an order ϵ change in φ (and therefore of kinetic energy) as the particle traverses the surface. This change in φ can be calculated directly from the product of the drift velocity normal to the surface and the electric field

$$\begin{aligned}\frac{d\varphi}{dt} &= 2\,\frac{\mathbf{e}_1}{B} \times \left[-\frac{mc}{e}\, v_\parallel\, \Omega \mathbf{e}_3 (\mathbf{e}_r \cdot \mathbf{e}_1) \right] \cdot (-\mathbf{E}) \\ &= \frac{2mc}{eB}\, v_\parallel \Omega\, \frac{\partial r}{\partial s}\, \mathbf{e}_1 \cdot (\mathbf{e}_3 \times \mathbf{E}) \\ &= -\frac{2mc}{eB}\, v_\parallel \Omega\, \frac{\partial r}{\partial s}\, E = -\frac{2m}{e}\, v_\parallel\, \Omega^2 r\, \frac{\partial r}{\partial s}.\end{aligned} \tag{34}$$

Or integrating

$$\Delta(e\varphi) = -m\Omega^2 \Delta(r^2) = -m\Delta(u_E^2). \tag{35}$$

The change in $e\varphi$ caused by the first-order drift off the surface equals twice the change in $(m/2)u_E^2$ as the particle moves in zero order on the surface. This result can also be obtained by energy conservation. The total average energy associated with the perpendicular motion is $MB + mu_E^2/2$. Therefore $(mv_\parallel^2/2) + MB + (mu_E^2/2) + e\varphi$ is a constant of the zero plus first-order motion. But from Eq. (31) $(m/2)v_\parallel^2 + MB - (m/2)u_E^2$ is a constant of the zero-order motion. By subtraction $\Delta(e\varphi) = -2\Delta(mu_E^2/2)$.

This drift normal to the flux surface is not cumulative, since the sign of $v_\parallel$ reverses when the particle reflects near the mirror.

E. The Current in a Collisionless Plasma

It will be proven in this section that the current density perpendicular to the magnetic field in a collisionless plasma is given by

$$\mathbf{j}_\perp = ne\dot{\mathbf{R}}_\perp + c(\nabla \times \mathfrak{M})_\perp, \tag{36}$$

where $\dot{\mathbf{R}}_\perp$ is the drift velocity at the point where $\mathbf{j}_\perp$ is required and is given by Eq. (17), n is the density of guiding centers, and $\mathfrak{M} = -nM\mathbf{e}_1$ is the magnetic moment per unit volume of particles having guiding centers at the required point. Equation (36) applies only to a single class of particle, that is, to particles which all have the same magnetic moment M and parallel velocity at the point where $\mathbf{j}_\perp$ is to be calculated. This is because $\dot{\mathbf{R}}_\perp$ at a given point is a function of M and $v_\parallel$. If several classes of particles are present, their currents can be superposed to give a total current

$$\mathbf{J}_\perp = Ne\bar{\dot{\mathbf{R}}}_\perp + c(\nabla \times \overline{\mathfrak{M}})_\perp, \tag{37}$$

where $\bar{\dot{\mathbf{R}}}_\perp$ and $\overline{\mathfrak{M}}$ are the average drift velocity and magnetic moment per unit volume. It is therefore only necessary to prove Eq. (36).

It may be argued that Eq. (36) requires no proof since the first term is the current due to the motion of the guiding centers, and the second term is the usual current due to the magnetization of a medium—i.e., due to the motion about the guiding center (see Longmire (*12*), for example, for a proof). However, it seems desirable to demonstrate formally the validity of Eq. (36), especially for the general case where there is a zero-order drift due to the electric field, and a nonsteady state.

The starting point in the proof is the Boltzmann equation, from which can be derived (Ref. *13*, p. 94) the macroscopic equation of motion

$$nm\, d\mathbf{V}/dt = -\nabla\cdot\mathbf{P} + ne\,(\mathbf{V}\times\mathbf{B}/c) + ne\mathbf{E}, \tag{38}$$

where $\mathbf{V} = \bar{\mathbf{v}}$ is the average particle velocity and $\mathbf{P}$ is the pressure tensor, given by $\mathbf{P} = nm\langle(\mathbf{v}-\mathbf{V})(\mathbf{v}-\mathbf{V})\rangle_{\mathrm{Av}}$. In Ref. *14* it is shown that for strong magnetic fields (i.e., for adiabatic particle motion) that $\mathbf{P}$ is diagonal with two of the diagonal components equal.

$$\mathbf{P} = P_{\parallel}\mathbf{e}_1\mathbf{e}_1 + P_{\perp}\,(\mathbf{e}_2\mathbf{e}_2 + \mathbf{e}_3\mathbf{e}_3), \tag{39}$$

where $P_{\parallel} = nm\langle(v_{\parallel} - V_{\parallel})^2\rangle_{\mathrm{Av}}$ and $P_{\perp} = (nm/2)\,\langle v_{\perp}^2\rangle = n\bar{M}B$ and $\mathbf{v}_{\perp}$ is the perpendicular velocity in the frame moving at $\mathbf{V}$. The divergence of $\mathbf{P}$ is given in Ref. *15* as

$$\nabla\cdot\mathbf{P} = \mathbf{e}_1\left[\frac{\partial P_{\parallel}}{\partial s} - \frac{(P_{\parallel} - P_{\perp})}{B}\frac{\partial B}{\partial s}\right] + \left[(P_{\parallel} - P_{\perp})\frac{\partial \mathbf{e}_1}{\partial s} + \nabla_{\perp}P_{\perp}\right], \tag{40}$$

where $\nabla_{\perp} = \mathbf{e}_2(\mathbf{e}_2\cdot\nabla) + \mathbf{e}_3(\mathbf{e}_3\cdot\nabla)$. Since only a single class of particles is to be considered here, $P_{\parallel} = 0$ and $P_{\perp} = (nm/2)\,v_{\perp}^2 = nMB$.

Next solve Eq. (38) for $\mathbf{V}$ by crossing it with $\mathbf{e}_1$:

$$\mathbf{V} = \mathbf{V}_{\parallel} + \frac{c\mathbf{e}_1\times\nabla\cdot\mathbf{P}}{neB} + \frac{c\mathbf{E}\times\mathbf{e}_1}{B} + \frac{mc}{eB^2}\,\mathbf{e}_1\times\frac{d\mathbf{V}}{dt}\,. \tag{41}$$

Since the last term has ϵ in the coefficient, an iterative procedure can be used and $d\mathbf{V}/dt$ replaced by its value for $\epsilon = 0$. When $\epsilon = 0$, the particle has zero radius of gyration, and the average velocity $\mathbf{V}$ is simply the zero-order guiding center motion $\mathbf{V} = \mathbf{e}_1 v_{\parallel} + \mathbf{u}_E$. With this expression for $\mathbf{V}$ and with Eq. (40) for $\nabla\cdot\mathbf{P}$ with $P_{\parallel} = 0$, Eq. (41) becomes

$$\mathbf{V} = \mathbf{V}_{\parallel} + \frac{c}{neB}\,\mathbf{e}_1\times\left(-P_{\perp}\frac{\partial\mathbf{e}_1}{\partial s} + \nabla_{\perp}P_{\perp}\right) + \frac{c\mathbf{E}\times\mathbf{e}_1}{B} + \frac{mc}{eB^2}\,\mathbf{e}_1\times\frac{d}{dt}\,(v_{\parallel}\mathbf{e}_1 + \mathbf{u}_E). \tag{42}$$

The expanded expression for $[d(v_{\parallel}\mathbf{e}_1 + \mathbf{u}_E)]/dt$ given in Eq. (16) is not needed here. The last two terms of Eq. (42) contain all guiding center drifts except the one due to ∇B, which is contained along with $(\nabla \times \mathfrak{M})_\perp$ in the second term:

$$\nabla \times \mathfrak{M} = -\nabla \times (Mn\mathbf{e}_1) = -Mn\,\nabla \times \mathbf{e}_1 + \mathbf{e}_1 \times \nabla(Mn)$$
$$= -(P_\perp/B)\nabla \times \mathbf{e}_1 + \mathbf{e}_1 \times \nabla(P_\perp/B) = -(P_\perp/B)\nabla \times \mathbf{e}_1 + \mathbf{e}_1 \times (1/B)\{\nabla P_\perp - (P_\perp/B)\nabla B\}$$
$$= -(P_\perp/B)[\mathbf{e}_1 \times (\partial \mathbf{e}_1/\partial s) + \mathbf{e}_2 \times (\mathbf{e}_2\cdot\nabla)\mathbf{e}_1 + \mathbf{e}_3 \times (\mathbf{e}_3\cdot\nabla)\mathbf{e}_1] + \mathbf{e}_1 \times \{(1/B)\nabla P_\perp - (P_\perp/B^2)\nabla B\}. \quad (43)$$

The second and third term in the square brackets are parallel to $\mathbf{e}_1$, as can be seen by taking their cross products with $\mathbf{e}_1$ and observing that $\mathbf{e}_1\cdot(\mathbf{e}_2\cdot\nabla)\mathbf{e}_1$ and $\mathbf{e}_1\cdot(\mathbf{e}_3\cdot\nabla)\mathbf{e}_1$ both are zero. Therefore from Eq. (43)

$$(c\mathbf{e}_1/B) \times [-P_\perp(\partial \mathbf{e}_1/\partial s) + \nabla_\perp P_\perp] = c(\nabla \times \mathfrak{M})_\perp + (cP_\perp/B)\mathbf{e}_1 \times \nabla B$$
$$= c(\nabla \times \mathfrak{M})_\perp + nMc\mathbf{e}_1 \times \nabla B \quad (44)$$

and Eq. (42) becomes, since $\mathbf{j} = ne\mathbf{V}$

$$\mathbf{j} = ne\mathbf{V}_\parallel + ne\,\dot{\mathbf{R}}_\perp + c(\nabla \times \mathfrak{M})_\perp = \mathbf{j}_\parallel + ne\,\dot{\mathbf{R}}_\perp + c(\nabla \times \mathfrak{M})_\perp,$$

which is the same as Eq. (36). This result has also been derived in Eq. (121) of Ref. *11* for the special case of static fields.

It is also true that $\mathbf{j}_\parallel = nev_\parallel\mathbf{e}_1 + c(\nabla \times \mathfrak{M})_\parallel$. The proof is not so simple as for the perpendicular current density. It is necessary to work with the Boltzmann equation itself, rather than from one of its moments such as Eq. (38). Since the rigorous proof is quite lengthy, it will not be given in the present paper. The proof actually gives:

$$\mathbf{J} = Ne\langle(\dot{\mathbf{R}}_\perp + \mathbf{e}_1 v_\parallel)\rangle_{\mathrm{Av}} + c(\nabla \times \overline{\mathfrak{M}}), \quad (45)$$

which contains both the parallel and the perpendicular components, but it is a very much more difficult demonstration of the perpendicular component than presented here. N is the total guiding center density.

From Eq. (45) it can be seen that the current density cannot be determined from a knowledge of the guiding center motion alone. However, the rate of charge accumulation due to the divergence of the current density *can* be found from the guiding center motion alone, since

$$\partial(Ne)/\partial t = -\nabla\cdot\mathbf{J} = -\nabla\cdot[Ne\langle(\dot{\mathbf{R}}_\perp + \mathbf{e}_1 v_\parallel)\rangle_{\mathrm{Av}}].$$

The fact that $\nabla\cdot\mathbf{J}$ can be determined from the guiding center drifts alone is used in the next example.

F. The Particle Drift Explanation of Helmholtz Instability of a Plasma

It is instructive to explain in terms of particle motion why a given plasma instability occurs. This has been done for Rayleigh–Taylor (i.e., gravitational) plasma instability (*16*) by Rosenbluth and Longmire (*5*). They have shown how the guiding center drifts result in a regenerative increase in amplitude of a small sinusoidal perturbation of the plasma-vacuum interface. The Helmholtz instability of such an interface has been studied by the author (*17*) via the hydromagetic equations, without a detailed analysis of the particle motions. The particle drift explanation will now be given for a somewhat simpler case than in Ref. *17*, where one "fluid" was a vacuum magnetic field and the other a plasma with a pressure. A simpler example of Helmholtz instability occurs when two identical pressureless plasmas with the opposite velocities $\mathbf{v}_0$ and $-\mathbf{v}_0$ are separated by a sharp boundary (Fig. 5). There must be an electric field $E_0 = (v_0/c)B_0$ which produces the flow. There is a uniform surface charge on the interface.

Suppose that the interface is perturbed sinusoidally with an amplitude A, a wave number l, and with the surface charge density unchanged. The electric field is then also perturbed in such a way that the flow remains parallel to the boundary. It can be verified that the solution of $\nabla \cdot \mathbf{E} = 0$ for the perturbed electric field $\mathbf{E}$ is (in the upper plasma)

$$\begin{aligned} E_x &= -E_0 A l e^{-ly} \cos lx + o(A^2), \\ E_y &= E_0(1 + A l e^{-ly} \sin lx) + o(A^2). \end{aligned} \tag{46}$$

This is the field due to a uniform charge on a surface bent sinusoidally. The terms of $o(A^2)$ are not needed. From Eq. (46) it follows that the drift $\mathbf{u}_E = c\mathbf{E} \times \mathbf{e}_1/B$ is parallel to the perturbed boundary. Now for this simple geometry, the guiding center drift Eq. (17) reduces to

$$\dot{\mathbf{R}}_\perp = \mathbf{u}_E + \frac{mc}{eB} \mathbf{e}_1 \times (\mathbf{u}_E \cdot \nabla \mathbf{u}_E + \partial \mathbf{u}_E/\partial t). \tag{47}$$

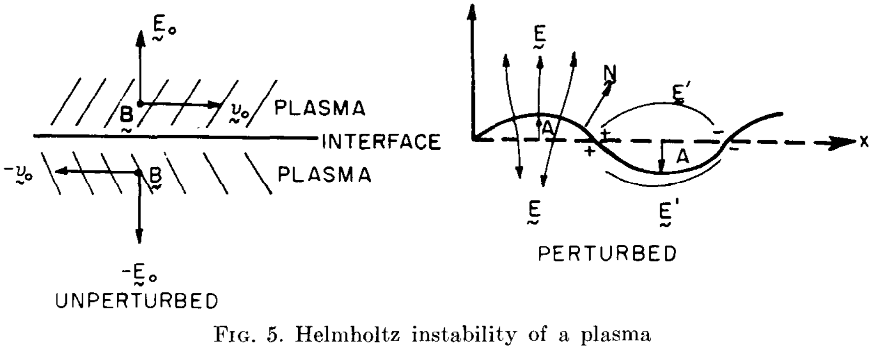

Fig. 5. Helmholtz instability of a plasma

There is no ∇B drift since with zero pressure $M = 0$. In the case of the Rayleigh instability there was a drift due to $\mathbf{g} \times \mathbf{B}$ which was opposite in direction for ions and electrons and therefore resulted in surface charge accumulation principally at the nodes of the perturbed surface. This charge accumulation resulted in an electric field which when crossed with $\mathbf{B}$, gave a drift that increased A. The increased amplitude in turn increased the rate of charge accumulation, hence the rate of increase of A, etc. For the present problem of Helmholtz instability, it is clear that the first term $\mathbf{u}_E$ in Eq. (47) cannot produce an accumulation of charge at the interface, since $\mathbf{u}_E = c\mathbf{E} \times \mathbf{e}_1/B$ is the same for ions and electrons. The mechanism of the instability must therefore be found in the second term where the direction of drift is opposite for ions and electrons. With the $\mathbf{E}$ given in Eq. (46) the drift at $y = 0$ due to the $\mathbf{u}_E \cdot \nabla \mathbf{u}_E$ term is

$$\frac{mc}{eB}\,\mathbf{e}_1 \times (\mathbf{u}_E \cdot \nabla)\mathbf{u}_E = \frac{mc}{e}\frac{Al^2E^2}{B^3}(\mathbf{x}\sin lx + \mathbf{y}\cos lx) + o(A^2), \qquad (48)$$

where $\mathbf{x}$ and $\mathbf{y}$ are unit vectors along the axes. The rate of free surface charge accumulation due to ions and electrons then is the component of guiding center current normal to the interface.

$$\begin{aligned}\partial\sigma/\partial t &= -2n\,|e|\,\mathbf{N}\cdot(m_i + m_e)(cAl^2E^2/|e|\,B^3)(\mathbf{x}\sin lx + \mathbf{y}\cos lx)\\ &= -2n(m_i + m_e)(cv_0^2l^2/B)\,A\cos lx + o(A^2),\end{aligned} \qquad (49)$$

where $\mathbf{N}$ is a unit vector normal to the interface, n is the ion or electron density, and m_i and m_e are ion and electron masses. The symmetry is such that the $\mathbf{u}_E \cdot \nabla \mathbf{u}_E$ drifts above and below the interface are additive in their effects on the surface charge, and this is the reason for the factor of 2. Since the drift which is producing the charge accumulation is first order in A, the normal vector $\mathbf{N}$ is needed only to zero order in A—i.e., $\mathbf{N} = \mathbf{y}$ is sufficient. The situation with Rayleigh instability is just the reverse: the drift which produces the charge accumulation is the $\mathbf{g} \times \mathbf{B}$ drift, which exists with the surface unperturbed and thus is of zero order in A. Consequently in the Rayleigh problem it is found that $\mathbf{N}$ must be used correct to first order in A.

It is now necessary to find the rate of increase of $\mathbf{E}'$, the field due to the surface charge accumulation. The dielectric constant

$$\kappa = 1 + [4\pi n\,(m_i + m_e)c^2/B^2] \cong 4\pi n\,(m_i + m_e)c^2/B^2$$

must be used, and also the fact that the displacement $\mathbf{D}$ has a continuous normal component and $\mathbf{E}'$ a continuous tangential component across the interface. The components of $\mathbf{E}'$ above the interface are easily found to be (to lowest order in A)

$$\begin{aligned}E_x' &= (2\pi\sigma e^{-ly}\sin lx)/\kappa\cos lx,\\ E_y' &= 2\pi\sigma e^{-ly}/\kappa.\end{aligned} \qquad (50)$$

The charge density from Eq. (49) is of the form $\sigma = \sigma_0 \cos lx$, so that E_x' does not diverge at $\cos lx = 0$. Since the field $\mathbf{E}$ gives motion parallel to the boundary, it is only $\mathbf{u}_{E'} = c\mathbf{E}' \times \mathbf{e}_1/B$ that makes A increase. At $y = 0$ and $lx = \pi/2$

$$dA/dt = c\mathbf{E}' \times \mathbf{e}_1/B = -cE_x'/B$$

or

$$\begin{aligned} d^2A/dt^2 &= -(c/B)(\partial E_x'/\partial t) = -(2\pi c/B\kappa)(d\sigma_0/dt) \\ &= (2\pi c^2/B^2\kappa)2n(m_i + m_e)v_0^2l^2A = v_0^2l^2A, \end{aligned} \tag{51}$$

Equation (51) is of the form $d^2A/dt^2 = \omega^2A$ where

$$\omega = v_0 l. \tag{52}$$

This is just the result that would be expected from Ref. *17* and the hydrodynamic expression in Lamb (*10*).

The drift term containing $\partial\mathbf{u}_E/\partial t$ in Eq. (47) equals $-c(\mathbf{e}_1/B) \times \partial(\mathbf{E} + \mathbf{E}')/\partial t$. Here the term $-c(\mathbf{e}_1/B) \times \partial\mathbf{E}/\partial t$ does not give rise to a charge accumulation at the interface; the surface charge density from this term due to plasma above the interface is proportional to $\sin lx$, while that due to plasma below is proportional to $(-\sin lx)$. If the two plasmas have the same densities n and opposite flow velocities, the proportionality constant is the same for $\sin lx$ and $(-\sin lx)$, with the result that there is no net surface charge. If the densities are unequal, then a net surface charge develops which gives rise to motion of the wave, hence to a complex ω.

The drift due to the other term $-c(\mathbf{e}_1/B) \times \partial\mathbf{E}'/\partial t$ has already been accounted for by use of the dielectric constant, since this is the term which gave the dielectric constant in the first example. Alternatively, κ can be set equal to unity in Eq. (50) and the drift due to $\partial\mathbf{E}'/\partial t$ retained. Differentiation of Eq. (50) then gives (at $y = 0$)

$$\partial\mathbf{E}'/\partial t = (2\pi/\cos lx)(\mathbf{x}\sin lx + \mathbf{y}\cos lx)\,\partial\sigma/\partial t. \tag{53}$$

If the $\partial\mathbf{E}'/\partial t$ drift is now added to Eq. (49) for $\partial\sigma/\partial t$ and $\partial\mathbf{E}'/\partial t$ eliminated via Eq. (53), an equation is obtained for $\partial\sigma/\partial t$. The solution for $\partial\sigma/\partial t$ is then substituted into Eq. (53), which then gives the same $\partial E_x'/\partial t$ as used in Eq. (51), hence the same ω as in Eq. (52).

G. An Example of the Drift Due to $\partial\mathbf{e}_1/\partial t$

In the preceding examples every drift in Eq. (17) has appeared with the exception of $v_\parallel(mc/eB)\mathbf{e}_1 \times \partial\mathbf{e}_1/\partial t$ and $(mc/eB)\mathbf{g} \times \mathbf{e}_1$. The latter occurs in a gravitational field and therefore is in principle present in every laboratory experiment and would also be exhibited by charged particles in the Van Allen radiation. In practice this gravitational drift is exceedingly small compared to the other drifts.

The well-known drifts proportional to ∇B and $v_\parallel^2\ \partial \mathbf{e}_1/\partial s$ were not discussed explicitly, but are present in the mirror machines as an azimuthal drift.

To illustrate the drift due to $\partial \mathbf{e}_1/\partial t$, consider a magnet with large parallel pole faces as shown in Fig. 6. Let the magnet be rotated about the Z-axis to give a $\partial \mathbf{e}_1/\partial t = \Omega \mathbf{y}$, where $\mathbf{y}$ is a unit vector along the Y axis and Ω is the magnet's angular velocity. Because there is a $\partial \mathbf{B}/\partial t$ there will in general be an $\mathbf{E}$, and therefore there will also occur the drift $\mathbf{u}_E = -c\mathbf{e}_1 \times \mathbf{E}/B$. Of the remaining terms in Eq. (17) the two proportional to $\mathbf{e}_1 \times \partial \mathbf{u}_E/\partial t$ and to $\mathbf{e}_1 \times \mathbf{u}_E \cdot \nabla \mathbf{u}_E$ are not obviously zero, although this will turn out to be the case. From $c\nabla \times \mathbf{E} = -\partial \mathbf{B}/\partial t$ one finds $\partial E_z/\partial x = \Omega B/c$ at zero time, if we take E_x and E_y zero. Then $E_z = \Omega B x/c + E_z(x = 0)$. Now $E_z(x = 0)$ also equals E_z everywhere for $\Omega = 0$. Let us assume there is no $\mathbf{E}$ in the absence of rotation, so that $E_z(x = 0)$ is zero. The $\mathbf{u}_E$ drift is

$$\mathbf{u}_E = c\mathbf{E} \times \mathbf{e}_1/B = x\Omega \mathbf{y} \text{ at zero time.} \tag{54}$$

Since $\mathbf{u}_E$ is independent of y, $\mathbf{u}_E \cdot \nabla \mathbf{u}_E = 0$. Also $\partial \mathbf{u}_E/\partial t$ is parallel to $\mathbf{e}_1$, so that $\mathbf{e}_1 \times \partial \mathbf{u}_E/\partial t = 0$. Thus both of these drift terms vanish leaving

$$\dot{\mathbf{R}}_\perp = c\mathbf{E} \times (\mathbf{e}_1/B) + v_\parallel (mc/eB)\mathbf{e}_1 \times \partial \mathbf{e}_1/\partial t = \mathbf{y}\Omega x + \mathbf{z} v_\parallel \Omega/\omega. \tag{55}$$

The $\partial \mathbf{e}/\partial t$ drift is perpendicular to the page and of magnitude $v_\parallel \Omega/\omega$.

The parallel equation of motion (20) becomes

$$dv_\parallel/dt = \mathbf{u}_E \cdot \partial \mathbf{e}_1/\partial t = \Omega^2 x. \tag{56}$$

This is just the centrifugal acceleration at a distance x from the axis of rotation.

The above example of a drift due to $\partial \mathbf{e}_1/\partial t$ does not explain geometrically why it occurs. All the drifts in Eq. (17) come from a variation at the gyration frequency of the curvature of the particle trajectory. This variation results in a cycloid-like motion. The reason for the variation at the gyration frequency is

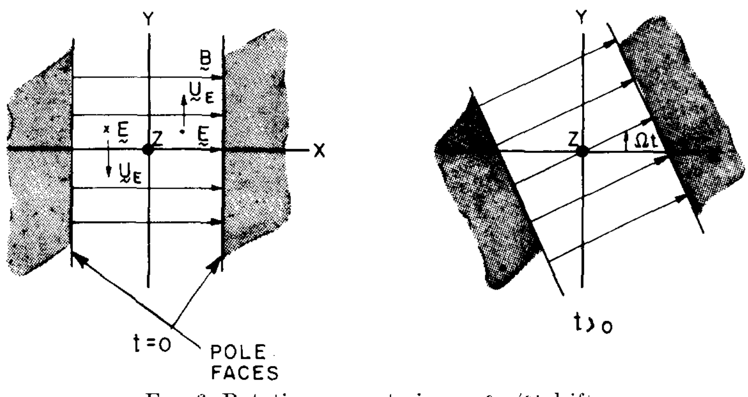

Fig. 6. Rotating magnet gives a $\partial \mathbf{e}_1/\partial t$ drift

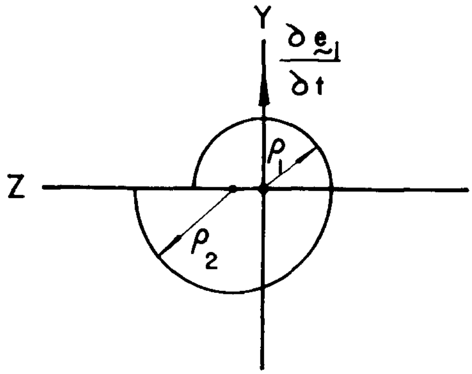

FIG. 7. Geometric explanation of the $\partial \mathbf{e}_1/\partial t$ drift

different for each drift. The more familiar drifts $\mathbf{u}_E$ and $(Mc/eB)\mathbf{e}_1 \times \nabla B$ have often been illustrated in the literature (*2*) and will not be discussed here. The reason that the curvature of the trajectory varies in the presence of $\partial \mathbf{e}_1/\partial t$ is that the perpendicular velocity $|\mathbf{v} \times \mathbf{e}_1|$ varies as $\mathbf{e}_1$ changes direction. The drift velocity can be derived (except possibly for a numerical factor) by holding $\mathbf{e}_1$ fixed for half a gyration period and then changing its direction for the next half period, etc. A view along the X axis of Fig. 6 at $t = 0$ will appear as in Fig. 7.

Let δ be the angle between $\mathbf{v}$ and $\mathbf{e}_1$, so that $v_\perp$ is $v \sin \delta$ and $v_\parallel$ is $v \cos \delta$. At the end of the first half period ($y > 0$) let $\mathbf{e}_1$ change by $\Delta \mathbf{e}_1$ in the y direction. For the second half period ($y < 0$) $v_\perp$ will be changed by $\Delta v_\perp = v \cos \delta \Delta \delta = v_\parallel \Delta \delta$. The drift velocity equals the difference in the diameters of the two semicircles divided by the gyration period, or $\omega(\rho_2 - \rho_1)/\pi$. Since ρ equals $v_\perp/\omega$, $\Delta\rho$ is $\Delta v_\perp/\omega$ or $v_\parallel \Delta\delta/\omega$. And $\Delta\delta = \Omega\pi/\omega$. Thus the drift velocity equals $v_\parallel \Omega/\omega$, which in this case happens to be correct even to numerical factors.

Similar geometric derivations can be given of the other drifts containing $v_\parallel$ and $\mathbf{u}_E$ in Eq. (17).

ACKNOWLEDGMENTS

The author is indebted to the many people whose work has been referred to in this paper. Special thanks are due Dr. Clifford Gardner, Dr. Allan Kaufman, and Dr. Marvin Mittleman for many helpful discussions and ideas.

RECEIVED: January 2, 1961

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